HOPF ALGEBRAS AND REPRESENTATIONS

(MS-E1993)

Practicalities

possibly subject to change!

Tue 14-16, M3, lecture
Thu 14-16, M3, lecture
Fri 12-14, M3, exercise session

5 credits

grading based on score
$\max\left(\tfrac{1}{2}\text{exam} + \tfrac{1}{2}\text{exercises}\ ,\ 1\cdot\text{exam}\right)$

exam: we will agree on a date during week December 12-16

exercises: Written exercise solutions to problems marked ✎ are to be returned to course TA Steven Flores by the beginning of each exerice session. Also the other problems are to be solved, and one should be prepared to present their solutions in the exercise session.

Literature:

lecture notes !

Kassel & Rosso & Turaev: "Quantum groups and knot invariants"

Fulton & Harris: "Representation theory: a first course"

Kassel: "Quantum groups"

All info: MyCourses ⇝ MS-E1993

Contents of the course

- representations of finite groups
- Hopf algebras
- braided Hopf algebras ("quantum groups")
  - constructions by Drinfeld double
  - example: the simplest quantum group $U_q(\mathfrak{sl}_2)$ and its representations
  - knot invariants from braided Hopf alg.

Introduction

"Representation" ≈ "linear action of a symmetry"

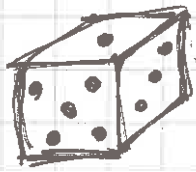
Why do we care about linear actions?

- often naturally the case, e.g.
  - functions on object of interest with values in $\mathbb{R}$ or $\mathbb{C}$ or ...
  - physical states in quantum mechanical systems are vectors in a Hilbert space
  - topological / geometric information encoded in vector spaces such as homology/cohomology
- rich and powerful theory with many applications!

A typical example application of rep. theory

~~Shuffling a deck of cards~~ too complicated, let's simplify :)

Shuffling cubic dice : which face is on top?



At time $t=0$, face ⚀ is on top.

At time $t+1$ apply a shuffling operation on the position of the die at time t
(for example: turn the die randomly on one of its four adjacent faces).

Markov chain $(X_t)_{t\in\mathbb{N}}$

$$X_t \in \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\} = F = \{\text{faces of cube}\}$$

Transition probabilities

$P_{a,b}$ $(a,b\in F)$ probability to turn from face a to face b.

After t "shufflings" distribution is

$$\mu_t(b) = \sum_{a_0,a_1,a_2,\ldots,a_{t-1}} \mu_0(a)\cdot P_{a,a_1} P_{a_1,a_2} \cdots P_{a_{t-2}a_{t-1}} P_{a_{t-1}b}$$

i.e. $\mu_t = \mu_0 \cdot P^t$.

μ_t is a function on F

$$\mu_t \in \mathbb{C}^F = \{ f: F \to \mathbb{C}\}$$ ← vector space

For large t, μ_t is governed by the leading eigenvectors of P.

Symmetry: P commutes with the action of the group of symmetries of the cube!

Representation theory gives the eigenvectors and multiplicities of different eigenvalues!

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}1\\0\\0\\0\\0\\-1\end{bmatrix} \text{"}\times 3\text{"} \qquad \begin{bmatrix}2\\-1\\-1\\-1\\-1\\2\end{bmatrix} \text{"}\times 2\text{"}$$

I REPRESENTATIONS OF FINITE GROUPS

BASIC DEFINITIONS AND BACKGROUND

<u>Def</u>. A <u>group</u> is a set G equipped with a binary operation $* : G \times G \longrightarrow G$ s.t.

- $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3 \quad \forall g_1, g_2, g_3 \in G$ "associativity"
- $\exists e \in G$ s.t. $\forall g \in G$
  $g * e = g = e * g$ "neutral element"
- $\forall g \in G \quad \exists g^{-1} \in G$ s.t.
  $g * g^{-1} = e = g^{-1} * g$ "inverse element"

<u>Remark</u> For abelian groups often $g_1 * g_2$ is denoted by $g_1 + g_2$ and for general groups we usually omit $*$ and denote $g_1 * g_2$ by $g_1 g_2$.

<u>Examples</u>
- <u>Cyclic group</u> $\mathbb{Z}/n\mathbb{Z}$ (abelian), $\#(\mathbb{Z}/n\mathbb{Z}) = n$.
- <u>Symmetric group</u> X a set "permutation group of X" "symmetric group on X"
  $S(X) = \{ \varphi : X \to X \text{ bijection} \}$
  group operation: composition of functions
  $\varphi_1 * \varphi_2 = \varphi_1 \circ \varphi_2 \qquad (\varphi_1 \circ \varphi_2 (x) = \varphi_1(\varphi_2(x)))$

  Special case: If $X = \{1, 2, \dots, n\}$ denote $S(X) = S_n$.
  $\# S_n = n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$
- <u>General linear group</u> $\mathbb{K}$ a field (say $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$)
  $GL_n(\mathbb{K}) = \{ M \in \mathbb{K}^{n \times n} \mid \det(M) \neq 0 \}$
  = the set of invertible $n \times n$ - matrices with entries in $\mathbb{K}$.
  group operation: matrix multiplication

Corresponding coordinate invariant description:

V a $\mathbb{K}$-vector space

$Aut(V) = \{ L : V \to V \text{ invertible linear map} \}$

group operation: composition of (linear) maps

If $\dim_{\mathbb{K}}(V) = n$ then $Aut(V) \cong GL_n(\mathbb{K})$.

- <u>Dihedral group</u>

  Symmetry group of regular n-gon

  generators r, m

  $r =$ rotation by $\frac{2\pi}{n}$ $\quad m =$ reflection

  relations $r^n = e$, $mrm = r^{-1}$, $m^2 = e$

  $G = \{ e, r, r^2, \dots, r^{n-1}, m, rm, r^2 m, \dots, r^{n-1} m \}$

  $$\# G = 2n$$

Terminology: the number of elements, $\# G$, is called the <u>order</u> of G.

G is said to be a finite group if $\# G < \infty$.

<u>Def</u>: If G and $\tilde{G}$ are two groups (group operations $*$ and $\tilde{*}$, respectively) then a mapping $f : G \to \tilde{G}$ is said to be a (group) <u>homomorphism</u> if $\forall g, h \in G$: $f(g * h) = f(g) \,\tilde{*}\, f(h)$.

<u>Examples</u>:
- Determinant $\det : GL_n(\mathbb{K}) \to \mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ is a homomorphism from general linear group to the multiplicative group of invertible scalars.
- The signature of a permutation (parity of the number of transpositions in any expression) $sgn : S_n \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism from the symmetric group to the two element group $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\} \subset \mathbb{R}^\times$

You should be familiar with notions of subgroup, normal subgroup, quotient group, kernel, isomorphism, ...

For any algebraic structure, one of the first basic results is the isomorphism theorem. For groups:

Thm (Isomorphism theorem for groups)

Let G, H be groups and $f: G \to H$ a homomorphism.
Then
1°) $\mathrm{Im}(f) = f(G) \subset H$ is a subgroup
2°) $\mathrm{Ker}(f) = f^{-1}(\{e_H\}) \subset G$ is a normal subgroup
3°) the quotient $G/\mathrm{Ker}(f)$ is isomorphic to $\mathrm{Im}(f)$.

It's good to realize that isomorphism theorems are similar for all algebraic structures, once the suitable notions have been given. In this course, for example, we will use the following cases:

STRUCTURE	MORPHISM f	IMAGE $\mathrm{Im}(f)$	KERNEL $\mathrm{Ker}(f)$ (OR FACTOR TO BE DIVIDED BY)
group	homomorphism	subgroup	normal subgroup
vector space	linear map	vector subspace	vector subspace
Lie algebra	(Lie algebra) homomorphism	Lie subalgebra	ideal
representation	intertwining map	subrepresentation	subrepresentation
⋮	⋮	⋮	⋮

We can summarize the isomorphism theorem in the commutative diagram:

if $f: G \to H$ is a group homomorphism, then there exists a unique group isomorphism $\tilde{f}: G/\mathrm{Ker}(f) \to \mathrm{Im}(f) \subset H$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\text{projection}"'] & & H \\ & G/\mathrm{Ker}(f) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Def.: An action of a group G on a set X is a group homomorphism
$$\alpha : G \longrightarrow S(X)$$
So, concretely, $g \in G$ acts on $x \in X$ by
$$x \longmapsto (\alpha(g))(x) =: g.x \qquad \text{(convenient notation)}$$
and the defining requirements of group action are
$$e.x = x \quad \forall x \in X \qquad \text{and} \qquad (gh).x = g.(h.x).$$

Examples

- Any group acts on itself by left multiplication
$$(\alpha(g))(h) = g * h \qquad (\text{or briefly } g.h = gh) \quad (g,h \in G)$$
- The symmetric group S_n acts on $\{1,2,\dots,n\}$ in the obvious way: $\sigma \in S_n$ $x \in \{1,\dots,n\}$: $\sigma.x = \sigma(x)$.
- The dihedral group acts on the set of vertices of an n-gon in the obvious way.

Def.: A representation of a group G on a vector space V is a group homomorphism
$$\rho : G \longrightarrow \mathrm{Aut}(V)$$
So, concretely, $g \in G$ acts by a linear map $\rho(g)$ on vectors $v \in V$
$$v \longmapsto \rho(g)\, v = g.v \qquad \text{(convenient notation — } V \text{ becomes a (left) } G\text{-module)}$$
and the defining requirements of a representation are
$$g.(v_1+v_2) = g.v_1 + g.v_2 \qquad \forall g \in G \quad \forall v_1, v_2 \in V$$
$$g.(\lambda v) = \lambda\, g.v \qquad \forall g \in G \quad \forall v \in V \quad \forall \lambda \in K.$$
$$e.v = v \qquad \forall v \in V$$
$$(gh).v = g.(h.v) \qquad \forall g,h \in G \quad \forall v \in V.$$

Def.: A subspace $V' \subset V$ of a representation $\rho : G \to \mathrm{Aut}(V)$ is a <u>subrepresentation</u> if $\forall g \in G$ $\rho(g) V' \subset V'$.

Then the restrictions $g \mapsto \rho(g)|_{V'}$ define a representation on V'.

<u>Remark</u> We almost always consider complex vector spaces!

<u>Examples</u>

- <u>trivial representation</u>

  $\rho(g) = \mathrm{id}_V \quad \forall g \in G$ trivial representation on V

  The case $V = \mathbb{C}$ with $\rho(g) = \mathrm{id}_{\mathbb{C}} \quad \forall g \in G$ is called <u>the trivial representation</u> of G.

- <u>fundamental representation</u> of <u>the dihedral group D_4</u>

  D_4: generators r, m, relations $r^4 = e$, $mrm = r^{-1}$

  Representation ρ on $V = \mathbb{R}^2$:

  $$r \mapsto R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad m \mapsto M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  Observation 0: It is sufficient to give the images of the generators.

  Observation 1: R and M satisfy the relations

  $$R^4 = \mathbb{1}_{2\times 2} \qquad M^2 = \mathbb{1}_{2\times 2} \qquad MRM = R^{-1}$$

  so indeed there exists a homom. $\rho: D_4 \to \mathrm{Aut}(\mathbb{R}^2)$ defined by these.

  Observation 2: One can concretely check that $\mathrm{Im}(\rho)$ consists of 8 different matrices. This is basically the easiest way to prove that $\# D_4 \geq 8$. (To show $\# D_4 \leq 8$ one just uses the relations to reduce any word to r^n or mr^n, $n = 0, 1, 2, 3$.)

REPRESENTATIONS AND THEIR EQUIVALENCES

Recall: A representation of a group G on a vector space V is a homomorphism
$$\rho : G \longrightarrow Aut(V) = \{T : V \to V \text{ bijective linear map}\}.$$

Convenient notation: $\rho(g)v = g.v$ for $g \in G$, $v \in V$

$\rightsquigarrow$ V becomes a left G-module

$\therefore$ representations of $G \longleftrightarrow$ left G-modules

Maps between representations which preserve the representation structure are called alternatively either intertwining maps or G-module maps.

<u>Def</u>: Let $\rho_1 : G \to Aut(V_1)$ and $\rho_2 : G \to Aut(V_2)$ be two representations of a group G. A linear map $f : V_1 \to V_2$ is called an <u>intertwining map</u> if
$$\forall g \in G : \quad f \circ \rho_1(g) = \rho_2(g) \circ f.$$

i.e. $\forall g \in G \ \forall v \in V \quad f(g.v) = g.f(v)$

A bijective intertwining map is called either an <u>isomorphism</u> or an <u>equivalence</u> (of representations).

One fundamental question in representation theory is:

- Can we classify all representations (possibly all finite-dim.) of a given group up to isomorphism?

As for other algebraic structures, one has an isomorphism theorem for representations (of a given G).

<u>Exercise</u> State and prove the isomorphism theorem for representations.

It is common to denote the space of intertwining maps between two representations V_1, V_2 of G by $Hom_G(V_1, V_2)$.

OPERATIONS ON REPRESENTATIONS

Recall: A representation of a group G is a homomorphism $\rho : G \longrightarrow \mathrm{Aut}(V)$ from G to the group of invertible linear maps of a vector space V.

We often denote briefly $\rho(g)v = g.v$ for $g \in G$, $v \in V$.

We will next show how to construct new representations from given ones by

- direct sums $V_1 \oplus V_2$
- tensor products $V_1 \otimes V_2$
- spaces of linear maps $\mathrm{Hom}(V_1, V_2)$ and in particular duals $V^* = \mathrm{Hom}(V, \mathbb{K})$
- invariants

In the following three constructions we assume that $\rho_1 : G \longrightarrow \mathrm{Aut}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{Aut}(V_2)$ are two representations.

Direct sum

Recall that $V_1 \oplus V_2$ is the vector space of pairs (v_1, v_2) of vectors $v_1 \in V_1$, $v_2 \in V_2$. (Usually denoted $v_1 + v_2$)

This becomes a representation by setting

$$\rho(g)\,(v_1, v_2) = (\rho_1(g)v_1 \,,\, \rho_2(g)v_2).$$

i.e. $g.(v_1 + v_2) = g.v_1 + g.v_2$ in the module notation.

Tensor product

Recall the construction of the tensor product vector space $V_1 \otimes V_2$ (spanned by $v_1 \otimes v_2$ with $v_1 \in V_1$, $v_2 \in V_2$)

This becomes a representation by setting

$$\rho(g)\,(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2$$

and extending linearly.

i.e. $g.(v_1 \otimes v_2) = g.v_1 \otimes g.v_2$

Linear maps

The space of linear maps from V_1 to V_2 is denoted by $\mathrm{Hom}(V_1, V_2)$ and it is itself a vector space.

This becomes a representation by setting, for any linear map $T: V_1 \to V_2$ and $g \in G$

$$\rho(g)\, T = \rho_2(g) \circ T \circ \rho_1(g^{-1}) : V_1 \to V_2$$

i.e. $(g.T)(v) = g.(T(g^{-1}.v)) \quad \forall v \in V_1$

Check: Clearly $\rho(g): \mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_1, V_2)$ is linear. We thus only need to observe that for any $g, h \in G$

$$\begin{aligned}
\rho(g)(\rho(h)T) &= \rho_2(g) \circ (\rho(h)T) \circ \rho_1(g^{-1}) \\
&= \rho_2(g) \circ \rho_2(h) \circ T \circ \rho_1(h^{-1}) \circ \rho_1(g^{-1}) \\
&= \rho_2(gh) \circ T \circ \rho_1((gh)^{-1}) \\
&= \rho(gh)\, T
\end{aligned}$$

or perhaps more concretely

$$\begin{aligned}
(g.(h.T))(v) &= g.((h.T)(g^{-1}.v)) \\
&= g.(h.(T(h^{-1}.(g^{-1}.v)))) \\
&= gh.\, T((gh)^{-1}.v) \\
&= ((gh).T)(v)
\end{aligned}$$

Duals

If $\rho: G \to \mathrm{Aut}(V)$ is a representation, then as a special case of the previous construction, the dual $V^* = \mathrm{Hom}(V, \mathbb{K}) = \{\varphi: V \to \mathbb{K} \text{ linear}\}$ becomes a representation. Note that on the 1-dimensional vector space $\mathbb{K}$ we use the trivial representation — all $g \in G$ act as identity on $\mathbb{K}$. Concretely, for $g \in G$, $\varphi \in V^*$, $v \in V$ we have

$$\langle g.\varphi, v \rangle = \langle \varphi, g^{-1}.v \rangle .$$

SUBREPRESENTATIONS, IRREDUCIBILITY AND COMPLETE REDUCIBILITY

Def: A *subrepresentation* of a representation $\rho : G \longrightarrow \mathrm{Aut}(V)$ is a vector subspace $\tilde{V} \subset V$ such that $\forall g \in G$: $\rho(g)\,\tilde{V} \subset \tilde{V}$. ("invariant subspace")
Note that $\tilde{V}$ is itself naturally a representation, with $\tilde{\rho} : G \to \mathrm{Aut}(\tilde{V})$ defined by restriction

$$\tilde{\rho}(g) = \rho(g)\big|_{\tilde{V}} \qquad \forall g \in G.$$

Examples

- The zero subspace $\{0\} \subset V$ and the full space $V \subset V$ are obviously always subrepresentations.
- If $V = V_1 \oplus V_2$ is the direct sum of two representations, then $V_1 \subset V$ and $V_2 \subset V$ are subrepresentations.

It turns out that the last example above is in fact the general case — any subrepresentation is a direct summand in the full representation (Well, we need the group G to be finite and the field $\mathbb{K}$ to have characteristic 0.)

Proposition: Let G be a finite group, and assume that the characteristic of the field $\mathbb{K}$ does not divide the order $\# G$ of the group.
If V is a representation of G, and $V' \subset V$ is a subrepresentation, then there exists another subrepresentation $V'' \subset V$ such that $V = V' \oplus V''$.

(This V'' is called a "complementary subrepresentation")

Proof: We can first choose a complementary vector subspace $U \subset V$, i.e., the equality
$$V = V' \oplus U$$
holds as vector spaces.

Let $\pi' : V \longrightarrow V'$ be the projection to first summand, i.e. $\pi'(v'+u) = v'$ if $v' \in V' \subset V$ and $u \in U \subset V$.

Define $\pi : V \longrightarrow V$ by
$$\pi(v) = \frac{1}{\# G} \sum_{g \in G} g.\pi'(g^{-1}.v).$$

(Recalling $\mathrm{Hom}(V,V)$ representation, this says $\pi = \frac{1}{\# G} \sum_{g} (g.\pi')$)

We will show that this π is a projection to V' as well.

First check that $\mathrm{Im}(\pi) \subset V'$:
recall that $\pi'(w) \in V'$ for any $w \in V$, and $g.V' \subset V'$ for all $g \in G$, so each term in the sum is in V'.

Then check that the restiction of π to V' is the identity: if $v' \in V' \subset V$ then $g^{-1}.v' \in V'$ and thus $\pi'(g^{-1}.v') = g^{-1}.v'$ and finally $g.\pi'(g^{-1}.v') = g.g^{-1}.v' = v'$, which shows that $\pi(v') = \frac{1}{\# G} \sum_{g \in G} v' = v'$.

The final observation is that π is an intertwining map: indeed, if $h \in G$, we have
$$\pi(h.v) = \frac{1}{\# G} \sum_{g \in G} g.\pi'(g^{-1}h.v)$$
(Change of variables $\tilde{g} = h^{-1}g$ in the sum)
$$= \frac{1}{\# G} \sum_{\tilde{g} \in G} (h\tilde{g}).\pi'(\tilde{g}^{-1}.v)$$
$$= \frac{1}{\# G} \sum_{\tilde{g} \in G} h.\left(\tilde{g}.\pi'(\tilde{g}^{-1}.v)\right) = h.\pi(v).$$

Now $V'' = \mathrm{Ker}(\pi)$ is a subrepresentation, and we have $V = V' \oplus V''$. $\square$

So whenever we find a subrepresentation, we are able to decompose the representation as a direct sum of two pieces. This motivates:

<u>Def</u>. A representation V is called <u>irreducible</u> if $V \neq \{0\}$ and V has no other subrepresentations but $\{0\}$ and V.

The idea is that we can decompose any representation to a direct sum of irreducible pieces.

<u>Theorem</u> Let G be a finite group, and assume that the characteristic of $\mathbb{K}$ does not divide $\#G$. Any finite dimensional representation V of G can be written as $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where $V_1, V_2, \dots, V_n$ are irreducible subrepresentations, and this decomposition is unique up to permutations of the summands.

<u>Proof</u>: Easy induction on $\dim(V)$. $\square$

The task of classifying of all (finite-dimensional) representations of G is thus reduced to the classification of all irreducible representations.

Schur's lemma

There is very little freedom for constructing intertwining maps between irreducible representations — and this turns out to be really crucial in all representation theory:

Theorem (Schur's lemma)

Let V and W be irreducible representations of a group G, and let $f: V \to W$ be an intertwining map. Then either $f \equiv 0$ or f is an isomorphism.

Proof: If $\operatorname{Ker}(f) \neq \{0\}$, then by irreducibility of V we have $\operatorname{Ker}(f) = V$ and so $f \equiv 0$.
If $\operatorname{Ker}(f) = \{0\}$ then f is injective, and so $\operatorname{Im}(f) \neq \{0\}$, and thus by irreducibility of W we have $\operatorname{Im}(f) = W$. $\square$

Let us now assume that $\mathbb{K}$ is algebraically closed (for most of this course we take $\mathbb{K} = \mathbb{C}$).
Then we can conclude:

Theorem (also called Schur's lemma).

Let V be an irreducible representation of G and $f: V \to V$ an intertwining map. Then we have $f = \lambda \cdot \mathrm{id}_V$ for some scalar $\lambda \in \mathbb{K}$.

Proof: Pick one eigenvalue λ of f. Then $f - \lambda \cdot \mathrm{id}_V$ is an intertwining map, and $\operatorname{Ker}(f - \lambda \cdot \mathrm{id}_V) \neq \{0\}$, so by irreducibility $\operatorname{Ker}(f - \lambda \cdot \mathrm{id}_V) = V$. $\square$

Corollary (also called Schur's lemma).

Let V and W be irreducible representations of G.
We have $\dim\left(\operatorname{Hom}_G(V, W)\right) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases}$

Invariants

Let G be a group, and $\rho : G \longrightarrow \mathrm{Aut}(V)$ a representation. Then the subspace (of "invariants")

$$V^G = \{ v \in V \mid \forall g \in G : \rho(g)v = v \} = \{ v \in V \mid \underset{\forall g \in G}{g.v = v} \}$$

is obviously a subrepresentation.

There is one particularly important case of this: the invariants of the space of linear maps between two representations.

Proposition $\underbrace{\mathrm{Hom}(V_1, V_2)^G}_{\text{invariants in the space of linear maps}} = \underbrace{\mathrm{Hom}_G(V_1, V_2)}_{\text{intertwining maps}}$

Proof: "$\supset$": Suppose $f : V_1 \to V_2$ is an intertwining map.

Then for any $g \in G$

$$(g.f)(v) \overset{\text{(def)}}{=} g.f(g^{-1}.v) \overset{\text{(intertw.)}}{=} g.g^{-1}.f(v) = f(v).$$

"$\subset$" Suppose $f : V_1 \longrightarrow V_2$ is an invariant in $\mathrm{Hom}(V_1, V_2)$.

Then for any $g \in G$ and all $w \in V_1$

$$g.f(g^{-1}.w) = f(w)$$

For a given $v \in V_1$ choose $w = g.v$ above to get $g.f(v) = f(g.v)$. $\square$

CHARACTER THEORY FOR REPRESENTATIONS OF FINITE GROUPS

Assume throughout this lecture:

- G a finite group
- $\mathbb{K} = \mathbb{C}$, all vector spaces are complex and linear maps complex-linear
- all representations of interest are finite-dimensional

Recall that under the above assumptions we have

- Any representation V of G is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ of irreducible subrepresentations $V_1, V_2, \ldots, V_n \subset V$. "complete reducibility"
- If V and W are irreducible representations of G which are not isomorphic to each other, then there are no non-zero intertwining maps between them
  "Schur's lemma, part 1"
- If V is an irreducible representation of G, then any intertwining map $V \to V$ is a scalar multiple of the identity, $\lambda \cdot \mathrm{id}_V$, $\lambda \in \mathbb{C}$
  "Schur's lemma, part 2"

The last two properties can be concisely summarized:

$$\dim\left(\mathrm{Hom}_G(V,W)\right) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W \end{cases}$$

for V, W irreducible.

In this lecture we will show that there are only finitely many different irreducible representations of G (isomorphic ones are identified) and we show how various questions about representations of G can be turned into straightforward calculations with characters.

Def: The <u>character</u> χ_ρ of a representation $\rho: G \longrightarrow \mathrm{Aut}(V)$ is the function $\chi_\rho : G \to \mathbb{C}$ given by $\chi_\rho(g) = \mathrm{tr}_V(\rho(g)) \quad \forall g \in G.$

Remarks:

- The trace tr_V is well defined, independent of the basis.
- If $g_1, g_2 \in G$ are conjugate, $g_2 = h g_1 h^{-1}$, then the character values are equal:

$$\chi_\rho(g_2) = \mathrm{tr}_V(\rho(g_2)) = \mathrm{tr}_V(\rho(h g_1 h^{-1}))$$
$$= \mathrm{tr}_V(\rho(h)\rho(g_1)\rho(h^{-1})) \underset{\substack{\uparrow \\ \text{cyclicity of trace:} \\ \mathrm{tr}(ABC) = \mathrm{tr}(BCA)}}{=} \mathrm{tr}_V(\rho(g_1))$$
$$= \chi_\rho(g_1)$$

  Therefore characters are constants on each conjugacy class of G.
  Such functions are called <u>class functions</u>.
- At the neutral element $e \in G$ we have
$$\chi_\rho(e) = \mathrm{tr}_V(\rho(e)) = \mathrm{tr}_V(\mathrm{id}_V) = \dim(V).$$
- The character of the trivial representation $\mathbb{C}$ ($\rho(g) = \mathrm{id}_\mathbb{C} \ \forall g \in G$) is constant one: $\chi_{\mathrm{triv.}}(g) = 1 \quad \forall g \in G.$

We next check what the operations that we can perform on representations (direct sum, tensor product, dual, etc.) do to characters.

First observe the following:

Lemma If G is a (finite) group, and $\rho: G \to \mathrm{Aut}(V)$ a (finite-dimensional complex) representation of G, then for each $g \in G$, the linear map $\rho(g): V \to V$ is diagonalizable, and all eigenvalues λ of $\rho(g)$ are roots of unity:
$$\lambda^n = 1 \quad \text{where} \quad n = \#G \text{ is the order of } G.$$

Proof: The order m of $g \in G$ divides the order $n = \#G$ of the group. Now since $g^m = e$, we have
$$\rho(g)^m = \rho(g^m) = \rho(e) = id_V.$$
Therefore the minimal polynomial of $\rho(g)$ divides the polynomial $x^m - 1$. But the roots of $x^m - 1$ are simple, so the roots of the minimal polynomial are also simple, and therefore $\rho(g)$ is diagonalizable.
It follows also that eigenvalues λ of $\rho(g)$ satisfy $\lambda^m = 1$ and since $m | n$, also $\lambda^n = 1$. $\square$

Theorem Let $\rho_V: G \to \mathrm{Aut}(V)$ and $\rho_W: G \to \mathrm{Aut}(W)$ be two representations of G, and χ_V and χ_W their characters. Then we have, $\forall g \in G$

- $\chi_{V^*}(g) = \overline{\chi_V(g)}$ (character of dual rep.)
- $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$ (character of direct sum rep.)
- $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$ (character of tensor product rep.)

Remark: By the Lemma, $\chi_\rho(g)$ is the sum of eigenvalues of $\rho(g): V \to V$ (with multiplicity).

Proof: Denote the dimensions by $n = \dim(V)$, $m = \dim(W)$.
Fix $g \in G$, and diagonalize $\rho_V(g) : V \to V$ by a basis $v_1, \ldots, v_n$ of eigenvectors, with respective eigenvalues $\lambda_1, \ldots, \lambda_n$, so that

$$\rho_V(g)\, v_i = \lambda_i v_i \qquad \forall i = 1, \ldots, n.$$

Let then $v_1^*, \ldots, v_n^*$ be the dual basis of V^*, so that

$$\langle v_j^*, v_i \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that by definition of the dual representation

$$\langle \rho_{V^*}(g)\, v_j^*, v_i \rangle = \langle v_j^*, \rho_V(g^{-1}) v_i \rangle = \lambda_i^{-1} \langle v_j^*, v_i \rangle = \lambda_i^{-1} \delta_{ij} = \lambda_j^{-1} \delta_{ij}$$

for all i, so we have

$$\rho_{V^*}(g)\, v_j^* = \lambda_j^{-1} v_j^*.$$

But recall that λ_j is a root of unity, so $\lambda_j^{-1} = \overline{\lambda_j}$. Therefore $v_1^*, \ldots, v_n^*$ is a basis of eigenvectors of $\rho_{V^*}(g)$, with respective eigenvalues $\overline{\lambda_1}, \ldots, \overline{\lambda_n}$. Now

$$\chi_{V^*}(g) = \overline{\lambda_1} + \cdots + \overline{\lambda_n} = \overline{(\lambda_1 + \cdots + \lambda_n)} = \overline{\chi_V(g)}.$$

Similarly, let $w_1, \ldots, w_m$ be a basis of eigenvectors of $\rho_W(g) : W \to W$, with resp. eigenvalues $\mu_1, \ldots, \mu_m$.
Now $v_1, \ldots, v_n, w_1, \ldots, w_m$ is a basis of eigenvectors of $\rho_{V \oplus W}(g)$ on $V \oplus W$, and we get

$$\chi_{V \oplus W}(g) = \lambda_1 + \cdots + \lambda_n + \mu_1 + \cdots + \mu_m = \chi_V(g) + \chi_W(g).$$

Similarly, $(v_i \otimes w_k)_{i=1,\ldots,n;\, k=1,\ldots,m}$ is a basis of $V \otimes W$ and

$$\rho_{V \otimes W}(g)\,(v_i \otimes w_k) = \rho_V(g) v_i \otimes \rho_W(g) w_k = \lambda_i \mu_k \; v_i \otimes w_k$$

so $\chi_{V \otimes W}(g) = \sum_{i=1}^{n} \sum_{k=1}^{m} \lambda_i \mu_k = (\sum_i \lambda_i)(\sum_k \mu_k) = \chi_V(g)\, \chi_W(g)$. $\square$

Aside: How to pick the invariants in a representation?

V a representation of G

$V^G = \{ v \in V \mid g.v = v \ \forall g \in G \}$ subrepr. of "invariants".

Define a linear map φ on V by

$$\varphi(v) = \frac{1}{\# G} \sum_{g \in G} g.v \qquad (v \in V).$$

Lemma: The map φ is a projection $V \to V^G$.

Proof: If $v \in V^G$ then $\varphi(v) = \frac{1}{\# G} \sum_{g \in G} v = v$, so $\varphi|_{V^G} = id_{V^G}$.

Let $h \in G$, $v \in V$. Then with change of var. $k = hg$

$$h.\varphi(v) = \frac{1}{\# G} \sum_{g \in G} hg.v = \frac{1}{\# G} \sum_{k \in G} k.v = \varphi(v),$$

so $Im(\varphi) \subset V^G$. $\square$

Corollary $\dim(V^G) = \frac{1}{\# G} \sum_{g \in G} \chi_V(g)$

Proof: Evaluate $tr(\varphi)$ either by projection property (LHS) or directly from definition. $\square$

Proposition For V and W representations of G we have $\dim(Hom_G(V,W)) = \frac{1}{\# G} \sum_{g \in G} \overline{\chi_V(g)} \cdot \chi_W(g)$

Proof: Recall that $Hom(V,W) \cong W \otimes V^*$

$\Rightarrow \chi_{Hom(V,W)}(g) = \chi_W(g) \overline{\chi_V(g)}$ by properties of characters.

Recall also that $Hom_G(V,W) = Hom(V,W)^G$.

Therefore we get from the previous corollary

$$\dim(Hom_G(V,W)) = \frac{1}{\# G} \sum_{g \in G} \chi_{Hom(V,W)}(g) = \frac{1}{\# G} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

$\square$

Let us agree to use the following natural inner product on the space of functions $G \to \mathbb{C}$:

$$(\varphi, \phi) = \frac{1}{\# G} \sum_{g \in G} \overline{\varphi(g)} \cdot \phi(g) .$$

Then we can rewrite: $\dim(\mathrm{Hom}_G(V, W)) = (\chi_V, \chi_W)$.

We also easily get the following powerful theorem:

<u>Theorem</u>

(i) If V and W are irreducible representations of G then $(\chi_V, \chi_W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W \end{cases}$.

(ii) The characters of (non-isomorphic) irreducible representations are linearly independent.

(iii) The number of different (isomorphism classes of) irreducible representations is at most the number of conjugacy classes of G.

<u>Proof</u>: (i) follows from the above observation with Schur's lemma.

(ii) follows from (i), which guarantees orthogonality of irreducible characters.

(iii) follows from linear independence (ii) and the dimension of the space of class functions. $\square$

We proceed with further consequences.

Fix the finite group G, and let $(W_\alpha)_{\alpha=1}^{s}$ be the (finite) collection of all mutually non-isomorphic irreducible representations of G.

Let V be a representation of G, and for all α denote by $m_\alpha \in \mathbb{Z}_{\geq 0}$ the multiplicity of W_α in the decomposition $V = \bigoplus_{\alpha=1}^{s} m_\alpha W_\alpha$ given by complete reducibility.

Theorem

(i) $m_\alpha = (\chi_{W_\alpha}, \chi_V)$ for all $\alpha = 1, \dots, s$.

(ii) The character χ_V determines V up to isomorphism.

(iii) We have $(\chi_V, \chi_V) = \sum_{\alpha=1}^{s} m_\alpha^2$.

(iv) V is irreducible if and only if $(\chi_V, \chi_V) = 1$.

Proof: The character of $V \cong \bigoplus_\alpha m_\alpha W_\alpha$ is $\chi_V = \sum_\alpha m_\alpha \chi_{W_\alpha}$. Now for any $\beta = 1, \dots, s$, note that

$$(\chi_{W_\beta}, \chi_V) = \sum_\alpha m_\alpha \underbrace{(\chi_{W_\beta}, \chi_{W_\alpha})}_{= \delta_{\alpha,\beta} \text{ by previous Thm}} = m_\beta ,$$

which proves (i).

By (i), the character χ_V determines the multiplicity of any irreducible in the decomposition of V, and therefore the isomorphism type of V.

The formula (iii) follows by calculating

$$(\chi_V, \chi_V) = \Big(\sum_\alpha m_\alpha \chi_{W_\alpha}, \sum_\beta m_\beta \chi_{W_\beta}\Big) = \sum_{\alpha,\beta} m_\alpha m_\beta \underbrace{(\chi_{W_\alpha}, \chi_{W_\beta})}_{= \delta_{\alpha\beta}}$$
$$= \sum_\alpha m_\alpha^2 ,$$

and this readily implies (iv). $\square$

<u>Example</u> : Let us consider as an example the symmetric group on three letters, S_3.

In the exercises, you have found the conjugacy classes of S_3 :

identity : $\{e\}$

transpositions : $\{(12), (13), (23)\}$

3-cycles : $\{(123), (132)\}$.

Let us consider the following three representations:

- trivial rep $U = \mathbb{C}$, $\rho_U(\sigma) = id_{\mathbb{C}} \quad \forall \sigma \in S_3$
- alternating rep. $U' = \mathbb{C}$, $\rho_{U'}(\sigma) = sgn(\sigma) \cdot id_{\mathbb{C}} \quad \forall \sigma \in S_3$
- a two-dimensional rep. V :

  By realizing that S_3 is isomorphic to the dihedral group D_3 of order 6, we translate the defining representation of D_3 to the following representation of S_3 :

$$\rho_V((12)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_V((123)) = \begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

The character values on representatives e, (12), (123) of the conjugacy classes are now

	e	(12)	(123)
χ_U	1	1	1
$\chi_{U'}$	1	-1	1
χ_V	2	0	-1

The regular representation and group algebra

The group G acts on itself by left multiplication, and from this action we can construct a representation $\mathbb{C}^G$ with a basis $(e_g)_{g\in G}$ by

$$h.e_g = e_{hg} \qquad \forall h\in G,\ g\in G.$$

We denote this representation by $\mathbb{C}[G]$, and call it the (left) <u>regular representation</u> of G.

In fact, $\mathbb{C}[G]$ becomes an algebra (over $\mathbb{C}$) with a bilinear product defined on the basis elements by

$$e_h\, e_g = e_{hg}.$$

It is easy to calculate the character of $\mathbb{C}[G]$, since in the basis $(e_g)_{g\in G}$ the action of $h\in G$ has no diagonal entries unless $h=e\in G$:

$$\chi_{\mathbb{C}[G]}(h) = \begin{cases} \# G & \text{if } h=e \\ 0 & \text{if } h\neq e. \end{cases}$$

The multiplicity m_α of W_α in the regular representation is

$$m_\alpha = (\chi_{W_\alpha}, \chi_{\mathbb{C}[G]}) = \frac{1}{\# G}\sum_{g\in G} \overline{\chi_{W_\alpha}(g)}\, \chi_{\mathbb{C}[G]}(g)$$

$$= \frac{1}{\# G}\overline{\chi_{W_\alpha}(e)}\cdot \# G = \dim(W_\alpha).$$

Thus any irreducible representation of G appears in the regular representation with multiplicity equal to its dimension:

$$\boxed{\mathbb{C}[G] \cong \bigoplus_\alpha \dim(W_\alpha)\cdot W_\alpha}$$

Equating in particular the dimensions of the two sides we find that $\boxed{\# G = \sum_\alpha \dim(W_\alpha)^2}$.

<u>Example</u> Consider the symmetric group on 4 letters, S_4. It has five conjugacy classes:

- neutral element
- trasposition
- three-cycles
- four-cycles
- products of two disjoint transpositions.

What can we say about representations of S_4?

We know there are at most five different irreducible representations. The trivial and alternating representations are both one-dimensional irreducible. The sum of squares formula above says that
$$\sum_{\alpha} \dim(W_\alpha)^2 = \# S_4 = 4! = 24.$$

Let's subtract the known contributions of trivial and alternating representations:
$$\sum_{\alpha \neq \text{triv}, \text{alt}} \dim(W_\alpha)^2 = 24 - 1^2 - 1^2 = 22$$
and observe that this sum has at most three terms in it. But 22 is not a square, and not a sum of two squares, so we know that there are still 3 other irreducibles. Moreover, the only way of expressing 22 as a sum of 3 squares is $22 = 3^2 + 3^2 + 2^2$, so we know that S_4 has exactly five irreducibles in total, and their dimensions are
$$\underbrace{1, \; 1,}_{\substack{\text{trivial}\\ \text{and alternating}}} \; \underbrace{2, \; 3, \; 3}_{\substack{\text{other}\\ \text{irreducibles}}}.$$

In the exercises you will use the group algebra to show that

<u>Theorem:</u> The number of different irreducible representations of G equals the number of conjugacy classes of G, and their characters form an orthonormal basis for the space of class functions on G.

This suggests that the information about the irreducible representations is concisely summarized in the "<u>character table</u>" of G, which tabulates the values of the characters of irreducible representations on the conjugacy classes, as we did for S_3 in an earlier example:

S_3 character table

	e	transpos.	3-cycle
trivial	1	1	1
altern.	1	-1	1
2-dim	2	0	-1

The rows of the character table are orthonormal w.r.t. the inner product on class functions

$$(\varphi, \phi) = \frac{1}{\# G} \sum_{g \in G} \overline{\varphi(g)}\, \phi(g) = \frac{1}{\# G} \sum_{\text{conj. cl. } C} (\# C) \cdot \overline{\varphi(C)}\, \phi(C).$$

The colums are also orthogonal with respect to the appropriate inner product: for any two conjugacy classes C and D of G we have

$$\sum_{\substack{\alpha \\ \text{irred. rep.}}} \overline{\chi_\alpha(C)}\, \chi_\alpha(D) = \begin{cases} \# G / \# C & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}$$

ALGEBRAS, COALGEBRAS, BIALGEBRAS AND HOPF ALGEBRAS

Algebras

Def: An algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space A equipped with a bilinear operation $*: A \times A \longrightarrow A$, $(a,b) \longmapsto a * b$ (product or multiplication) and a distinguished element $1_A \in A$ (unit) such that:

associativity: $(a*b)*c = a*(b*c) \quad \forall a,b,c \in A$

unitality: $a * 1_A = a = 1_A * a \quad \forall a \in A$.

Examples:

1°) For V a $\mathbb{K}$-vector space, the space

$$\begin{aligned} \mathrm{End}(V) &= \mathrm{Hom}(V,V) \\ &= \{\mathbb{K}\text{-linear maps } V \to V\} \end{aligned}$$

is an algebra: the product is the composition of linear maps $(a,b) \longmapsto a \circ b$ and the unit is the identity map id_V.

Essentially equivalently (when $\dim(V) = d < \infty$) the space $\mathbb{K}^{d \times d}$ of $d \times d$ matrices with entries in $\mathbb{K}$ is an algebra: the product is matrix multiplication and the unit is the unit matrix $\mathbb{1}_{d \times d}$.

2°) The space
$$\mathbb{K}[t] = \{ c_n t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0 \mid n \in \mathbb{N},\ c_0, c_1, \dots, c_n \in \mathbb{K} \}$$
of polynomials with coefficients in $\mathbb{K}$ is an algebra: the product is the multiplication of polynomials and the unit is the constant polynomial 1.

3°) The group algebra $\mathbb{K}[G]$ of a group G is the vector space with basis u_g, $g \in G$. It is an algebra with the product obtained by bilinearly extending $u_g u_h = u_{gh}$ and with unit u_e, where $e \in G$ is the neutral element.

<u>Def</u>: An algebra A is commutative if $ab = ba \quad \forall a, b \in A$.

<u>Examples</u>: 1°) $\mathrm{End}(V)$ is not commutative if $\dim(V) \geq 2$.

2°) $\mathbb{K}[t]$ is commutative.

3°) $\mathbb{K}[G]$ is only commutative if the group G is abelian.

<u>Def</u>: An element $a \in A$ is invertible if there exists an element $a^{-1} \in A$ s.t. $a * a^{-1} = 1_A = a^{-1} * a$.

<u>Example</u>: 1°) In $A = \mathbb{K}^{d \times d}$ the invertible elements are the matrices M with $\det(M) \neq 0$.

2°) In $A = \mathbb{K}[t]$, the invertible elements are the non-zero constant polynomials $p(t) = p_0 \in \mathbb{K} \setminus \{0\}$

Def: If A and $\tilde{A}$ are algebras, with products $*$ and $\tilde{*}$, and units 1_A and $1_{\tilde{A}}$, respectively, then a linear map
$$f: A \to \tilde{A}$$
is said to be a homomorphism (of algebras) if $f(1_A) = 1_{\tilde{A}}$ and
$$f(a * b) = f(a) \,\tilde{*}\, f(b) \qquad \forall a, b \in A.$$
A bijective homomorphism is called an isomorphism.

Def: A subalgebra of an algebra A is a vector subspace $A' \subset A$ such that $1_A \in A'$ and $a' * b' \in A'$ $\forall a', b' \in A'$.

An ideal (two-sided) of an algebra A is a vector subspace $J \subset A$ such that for all $a \in A$ and $j \in J$ we have $a * j \in J$ and $j * a \in J$.

Proposition (Quotient algebra) If A is an algebra and $J \subset A$ is an ideal, then the quotient vector space A/J becomes an algebra with product $(a + J) * (b + J) = ab + J$ and unit $1_A + J$.

Theorem (Isomorphism theorem for algebras)
Let $f: A \to \tilde{A}$ be a homomorphism of algebras. Then:
(i) $\mathrm{Im}(f) \subset \tilde{A}$ is a subalgebra
(ii) $\mathrm{Ker}(f) \subset A$ is an ideal
(iii) $A/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$.

Example Consider the polynomial algebra $\mathbb{K}[t]$. Let $q(t) = q_d t^d + q_{d-1} t^{d-1} + \dots + q_1 t + q_0 \in \mathbb{K}[t]$ be a given polynomial and

$$\langle q(t) \rangle = \{ p(t)\, q(t) \mid p(t) \in \mathbb{K}[t] \}.$$

Then $\langle q(t) \rangle$ is an ideal and one may define the quotient $\mathbb{K}[t] / \langle q(t) \rangle$, which is a d-dimensional algebra.

Representations of algebras

Def: A representation of an algebra A on a vector space V is a homomorphism

$$\varrho : A \longrightarrow \mathrm{End}(V).$$

For $a \in A$, $v \in V$, we usually again denote

$$(\varrho(a))(v) =: a.v \in V.$$

The homomorphism property says that $a.v$ is linear in both a and v, and

$$(a * b).v = a.(b.v) \quad \text{and} \quad 1_A . v = v$$

for all $a, b \in A$, $v \in V$.
We also call V a left A-module, then, because elements of V can be "multiplied from the left" by elements of A.

Examples: 1°) The vector space V is naturally a representation of the algebra $A = \mathrm{End}(V)$.
(via $\varrho = \mathrm{id}_A$)

2°) For any matrix $M \in \mathbb{K}^{d \times d}$, the space $V = \mathbb{K}^d$ becomes a representation of the polynomial algebra $A = \mathbb{K}[t]$ by

$$\rho(c_n t^n + \dots + c_1 t + c_0) = c_n M^n + \dots + c_1 M + c_0 \mathbb{1}.$$

3°) Continuing with the previous example, the homomorphism ρ can be factored through the quotient algebra $\mathbb{K}[t]/\langle q(t) \rangle$ if and only if $q_m M^m + q_{m-1} M^{m-1} + \dots + q_0 \mathbb{1} = 0$, i.e. if the minimal polynomial of M divides $q(t)$. Then the factorization

$$\begin{tikzcd} \mathbb{C}[t] \arrow[r, "\rho"] \arrow[d, "\text{projection}"'] & \mathrm{End}(\mathbb{K}^d) \\ \mathbb{C}[t]/\langle q(t)\rangle \arrow[ur, dotted, "\tilde{\rho}"'] & \end{tikzcd}$$

defines a representation $\tilde{\rho}$ of the algebra $A = \mathbb{C}[t]/\langle q(t)\rangle$ on $\mathbb{K}^d$.

4°) If $\rho_{grp} : G \to \mathrm{Aut}(V)$ is a representation of a group G, then the linear extension of

$$\rho(u_g) = \rho_{grp}(g) \in \mathrm{Aut}(V) \subset \mathrm{End}(V)$$

defines a representation $\rho : \mathbb{K}[G] \to \mathrm{End}(V)$ of the group algebra. Vice versa also any representation of the group algebra $\mathbb{K}[G]$ restricts to a representation of the group G, so representations of G and $\mathbb{C}[G]$ can be used interchangeably.

<u>Def</u>: For a representation $\rho : A \to \mathrm{End}(V)$, a vector subspace $V' \subset V$ is called <u>invariant</u> if $\rho(a) V' \subset V'$ for all $a \in A$. Then the restriction $a \mapsto \rho(a)|_{V'}$ makes V' a <u>subrepresentation</u>.

Let us give two more examples of representations for a general algebra A.

Examples

5°) The algebra A is in particular a vector space, and becomes a representation of itself via left multiplication as follows.
(This generalizes the regular representation $\mathbb{C}[G]$ of a group G.)

For $a \in A$ define $\rho(a) \in \mathrm{End}(A)$ by $\rho(a)\, b = a * b$. Then $\rho : A \to \mathrm{End}(A)$ is a homomorphism because

- $\rho(1_A)\, b = 1_A * b = b \implies \rho(1_A) = \mathrm{id}_A$
- $\rho(a * a')\, b = (a * a') * b$
  $= a * (a' * b) = \rho(a)\left(\rho(a')\, b\right) \implies \rho(a * a') = \rho(a) \circ \rho(a')$.

This makes A (vect. sp.) a repr. of A (algebra).

6°) The dual $A^* = \mathrm{Hom}(A, \mathbb{K})$ becomes a representation of A using right multiplication.
For $\phi \in A^*$ and $a \in A$ define $a.\phi \in A^*$ by $\langle a.\phi, b \rangle = \langle \phi, b * a \rangle \quad \forall b \in A$.
The properties to check are:

- $\langle 1_A . \phi, b \rangle = \langle \phi, b * 1_A \rangle = \langle \phi, b \rangle \implies 1_A . \phi = \phi$
- $\langle (a * a').\phi, b \rangle = \langle \phi, b * (a * a') \rangle$
  $= \langle \phi, (b * a) * a' \rangle = \langle a'.\phi, b * a \rangle$
  $= \langle a.(a'.\phi), b \rangle \implies (a * a').\phi = a.(a'.\phi)$.

Operations on representations of algebras?

For representations $\rho_V : A \to \mathrm{End}(V)$, $\rho_W : A \to \mathrm{End}(W)$, can we perform operations like in the case of representations of groups? In other words, can we equip the following vector spaces with the structure of a representation?

(i) direct sum $V \oplus W$

(ii) trivial representation $\mathbb{K}$

(iii) tensor product $V \otimes W$

(iv) dual V^*

(v) linear maps $\mathrm{Hom}(V, W)$

(vi) Invariants? what does this even mean for algebras? we first need the notion of a trivial representation.

The first part (i) is clear:

Def: The direct sum representation of V and W is $\rho : A \longrightarrow \mathrm{End}(V \oplus W)$ given by

$$\rho(a)\,(v+w) = \rho_V(a)\,v + \rho_W(a)\,w \qquad \forall v \in V \;\; \forall w \in W.$$

The other operations turn out to require some additional structure for the algebra A. In other words, there is no canonical way to talk about trivial representations, tensor product representations, dual representations etc. for algebras in general.

Illustration: Consider e.g. the algebra $A = \mathbb{C}^{2\times 2}$ of 2×2 complex matrices. A trivial representation of it should equip $\mathbb{C}$ with the action of A in some natural way. What way?

Another illustration: If we think of the indeterminate t of a polynomial algebra $\mathbb{C}[t]$ as a derivation (action as a vector field), then for representations V and W of $\mathbb{C}[t]$ the tensor product should be defined using the Leibniz rule of differentiation

$$t.(v \otimes w) = (t.v) \otimes w + v \otimes (t.w)$$

or in more careful notation

$$\rho(t) = \rho_V(t) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \rho_W(t)$$
$$\in \mathrm{End}(V \otimes W)$$

This is very different from the group action on tensor products, where

$$g.(v \otimes w) = (g.v) \otimes (g.w) \qquad (g \in G)$$

or more carefully with the group algebra $\mathbb{C}[G]$

$$\rho(g) = \rho_V(g) \otimes \rho_W(g)$$
$$\in \mathrm{End}(V \otimes W)$$

Clearly both choices are interesting and relevant, so in general there must be some extra structure on A (e.g. $A = \mathbb{C}[t]$ or $A = \mathbb{C}[G]$) that dictates the choice!

Let us start with the trivial representation $\mathbb{K}$. In order to equip $\mathbb{K}$ with the structure of a representation of A, we need a homomorphism $A \longrightarrow \mathrm{End}(\mathbb{K})$ of algebras. But $\mathrm{End}(\mathbb{K}) \cong \mathbb{K}$ as algebras (the linear maps $\mathbb{K} \to \mathbb{K}$ are multiplications by a scalar $\lambda \in \mathbb{K}$), so this amounts to

$$\epsilon : A \longrightarrow \mathbb{K} \qquad \text{algebra homomorphism.}$$

(Note: $\mathbb{K}$ is obviously a $\mathbb{K}$-algebra with its own multiplication).

How about tensor products then? We would like to set, for any $a \in A$,

$$\rho_{V\otimes W}(a) = \sum_{j=1}^{m} \rho_V(a_j^{(1)}) \otimes \rho_W(a_j^{(2)})$$

for some $a_j^{(1)}, a_j^{(2)} \in A$, $j=1,2,\ldots,m$. Noting that the RHS must depend linearly on $a \in A$, we thus seek a linear map

$$\Delta : A \longrightarrow A \otimes A$$
$$a \longmapsto \sum_j a_j^{(1)} \otimes a_j^{(2)} .$$

Moreover, $\rho_{V\otimes W}$ needs to respect products and unit in A in order to become a homomorphism $A \longrightarrow \mathrm{End}(V \otimes W)$, so we want

$$\Delta : A \longrightarrow A \otimes A \quad \text{an algebra homomorphism.}$$

($A \otimes A$ is an algebra with product
$$(a_1 \otimes a_2) * (b_1 \otimes b_2) = (a_1 * b_1) \otimes (a_2 * b_2)$$
extended bilinearly and unit $1_{A\otimes A} = 1_A \otimes 1_A$.)

In other words, for the construction of trivial representations and tensor product representations, we want to equip the algebra A with two new structural maps, algebra homomorphisms

$$\epsilon : A \longrightarrow \mathbb{K}$$
$$\Delta : A \longrightarrow A \otimes A .$$

But do the tensor products and trivial representations have all the properties that we would like them to have?

We would like the "obvious" identities

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

$$\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$$

to hold as isomorphisms of representations (not just vector spaces). But as Δ only allows us to construct a tensor product of two representations, we need an iteration to construct multiple tensor products, and the two possible placements of parentheses correspond to iterations

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\Delta \otimes id_A} (A \otimes A) \otimes A \cong A \otimes A \otimes A$$
$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{id_A \otimes \Delta} A \otimes (A \otimes A) \cong A \otimes A \otimes A.$$

Similarly, constructing the representations $\mathbb{K} \otimes V$ and $V \otimes \mathbb{K}$ involve respectively

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\epsilon \otimes id_A} \mathbb{K} \otimes A \cong A$$
$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{id_A \otimes \epsilon} A \otimes \mathbb{K} \cong A.$$

In order to guarantee that the "obvious" identities of representations hold, we will require

(H1'): $\quad (\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta \quad : A \to A \otimes A \otimes A$

(H2'): $\quad (\epsilon \otimes id_A) \circ \Delta \cong id_A \cong (id_A \otimes \epsilon) \circ \Delta \quad : A \to A$

which can also be summarized as commutative diagrams

(H1'):
$$\begin{tikzcd} & A \otimes A \arrow[dr, "\Delta \otimes id"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A \otimes A \otimes A \\ & A \otimes A \arrow[ur, "id \otimes \Delta"'] & \end{tikzcd}$$

(H2'):
$$\begin{tikzcd} & A \otimes A \arrow[dl, "\epsilon \otimes id"'] \arrow[dr, "id \otimes \epsilon"] & \\ \mathbb{K} \otimes A & A \arrow[u, "\Delta"] \arrow[l, phantom, "\cong"] \arrow[r, phantom, "\cong"] & A \otimes \mathbb{K} \end{tikzcd}$$

Proposition Suppose A is an algebra, $\epsilon : A \to \mathbb{K}$ and $\Delta : A \to A \otimes A$ homomorphisms such that (H1') and (H2') hold. Equip $\mathbb{K}$ with the structure of representation of A using ϵ and tensor products of representations the structure of representation of A using Δ. Then the natural vector space isomorphisms

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

$$\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$$

are in fact isomorphisms of representations of A.

Proof: Left for the reader. $\square$

We finally emphasize the similarity of properties (H1') and (H2') with the defining properties of algebra: associativity and unitality.

For an algebra A we can define linear maps

$$\mu : A \otimes A \to A \qquad \mu(a \otimes b) = a * b$$

$$\eta : \mathbb{K} \to A \qquad \eta(\lambda) = \lambda \cdot 1_A .$$

Associativity and unitality amount to

(H1): $\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu)$ $\boxed{(a*b)*c = a*(b*c)}$

(H2): $\mu \circ (\eta \otimes id_A) \cong id_A \cong \mu \circ (id_A \otimes \eta)$ $\boxed{1_A * a = a = a * 1_A}$

summarized in commutative diagrams

(H1):

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\mu"] & \\ A \otimes A \otimes A \arrow[ur, "\mu \otimes id"] \arrow[dr, "id \otimes \mu"'] & & A \\ & A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(H2):

$$\begin{tikzcd} & A \otimes A \arrow[d, "\mu"] & \\ \mathbb{K} \otimes A \arrow[ur, "\eta \otimes id"] \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\cong"] & A \otimes \mathbb{K} \arrow[ul, "id \otimes \eta"'] \end{tikzcd}$$

Recall a few commutative diagrams from last time:

$$\begin{aligned} \mu &: A\otimes A \longrightarrow A && \text{"product"} \\ \eta &: \mathbb{K} \longrightarrow A && \text{"unit"} \\ \Delta &: A \longrightarrow A\otimes A && \text{"coproduct"} \\ \epsilon &: A \longrightarrow \mathbb{K} && \text{"counit"} \end{aligned}$$

(H1): "associativity"

$$\begin{tikzcd} & A\otimes A \arrow[dr, "\mu"] & \\ A\otimes A\otimes A \arrow[ur, "\mu\otimes id"] \arrow[dr, "id\otimes\mu"'] & & A \\ & A\otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(H1'): "coassociativity"

$$\begin{tikzcd} & A\otimes A \arrow[dl, "\Delta\otimes id"'] & \\ A\otimes A\otimes A & & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ & A\otimes A \arrow[ul, "id\otimes\Delta"] & \end{tikzcd}$$

(H2): "unitality"

$$\begin{tikzcd} \mathbb{K}\otimes A \arrow[r, phantom, "\cong"] \arrow[dr, "\eta\otimes id"'] & A \arrow[r, phantom, "\cong"] & A\otimes\mathbb{K} \arrow[dl, "id\otimes\eta"] \\ & A\otimes A \arrow[u, "\mu"] & \end{tikzcd}$$

(H2'): "counitality"

$$\begin{tikzcd} \mathbb{K}\otimes A \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\cong"] \arrow[d, "\Delta"] & A\otimes\mathbb{K} \\ & A\otimes A \arrow[ul, "\epsilon\otimes id"] \arrow[ur, "id\otimes\epsilon"'] & \end{tikzcd}$$

Purely in terms of equalities of maps, these read:

(H1): $\mu\circ(\mu\otimes id_A) = \mu\circ(id_A\otimes\mu) : A\otimes A\otimes A \to A$

(H1'): $(\Delta\otimes id_A)\circ\Delta = (id_A\otimes\Delta)\circ\Delta : A \to A\otimes A\otimes A$

(H2): $\underbrace{\mu\circ(\eta\otimes id_A)}_{A\cong\mathbb{K}\otimes A\to A} \cong \underbrace{id_A}_{A\to A} \cong \underbrace{\mu\circ(id_A\otimes\eta)}_{A\cong A\otimes\mathbb{K}\to A} : A \to A$

(H2'): $\underbrace{(\epsilon\otimes id_A)\circ\Delta}_{A\to\mathbb{K}\otimes A\cong A} \cong \underbrace{id_A}_{A\to A} \cong \underbrace{(id_A\otimes\epsilon)\circ\Delta}_{A\to A\otimes\mathbb{K}\cong A} : A\to A$

In terms of these, we define:

Def: An algebra (over $\mathbb{K}$) is a triple $\mathcal{A} = (A, \mu, \eta)$ where

- A is a $\mathbb{K}$-vector space
- $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are linear maps that satisfy the axioms (H1) and (H2).

Def: A coalgebra (over $\mathbb{K}$) is a triple $\mathcal{A} = (A, \Delta, \epsilon)$ where

- A is a $\mathbb{K}$-vector space
- $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{K}$ are linear maps that satisfy the axioms (H1') and (H2').

Def: A bialgebra (over $\mathbb{K}$) is a quintuple $\mathcal{A} = (A, \mu, \eta, \Delta, \epsilon)$ where

- A is a $\mathbb{K}$-vector space
- $\mu : A \otimes A \to A$, $\eta : \mathbb{K} \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to \mathbb{K}$ are linear maps that satisfy axioms (H1), (H1'), (H2), (H2') and moreover Δ and ϵ are homomorphisms of algebras

Recall: In view of the above, a bialgebra is just an algebra equipped with two homomorphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{K}$, satisfying (H1'), (H2'). These can be used to define **tensor product representations** (by Δ) and the **trivial repr.** (by ε).

Axioms (H1') and (H2') guarantee that the "obvious" isomorphisms

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

$$\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$$

are isomorphisms of representations.

<u>Example</u> (Matrix product coalgebra)

Let $n \in \mathbb{N}^*$ and let C be the $\mathbb{K}$-vector space with basis $(e_{ij})_{i,j\in\{1,2,\dots,n\}}$. ($\dim C = n^2$). Define $\Delta : C \longrightarrow C \otimes C$ by linear extension of

$$\Delta(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj}$$

and $\varepsilon : C \longrightarrow \mathbb{K}$ by linear extension of

$$\varepsilon(e_{ij}) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then (C, Δ, ε) is a coalgebra.

Check:

- coassociativity:

$$((\Delta \otimes id) \circ \Delta)(e_{ij}) = \sum_{k=1}^{n} \Delta(e_{ik}) \otimes e_{kj}$$

$$= \sum_{k,l=1}^{n} (e_{il} \otimes e_{lk}) \otimes e_{kj}$$

$$((id \otimes \Delta) \circ \Delta)(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes \Delta(e_{kj})$$

$$= \sum_{k,l=1}^{n} e_{ik} \otimes (e_{kl} \otimes e_{lj}).$$

These expressions are equal: just interchange the dummy variables k and l.

<u>Example</u> (The incidence coalgebra of a poset)

Let $(P, \preceq)$ be a poset (partially ordered set), i.e. P is a set and $\preceq$ is a binary relation on P which satisfies

$$p \preceq p \quad \forall p \in P \qquad (\text{"reflexivity"})$$

$$p \preceq q \text{ and } q \preceq r \Rightarrow p \preceq r \quad \forall p,q,r \in P. \qquad (\text{"transitivity"})$$

$$p \preceq q \text{ and } q \preceq p \Rightarrow p = q \quad \forall p,q \in P \qquad (\text{"antisymmetry"}).$$

For two elements $p, q \in P$ such that $p \leq q$, the interval from p to q is

$$[p, q] := \{ r \in P \mid p \leq r,\ r \leq q \}.$$

The set of intervals is denoted I_P.

Let C_P be the vector space with basis $(e_{[p,q]})_{[p,q] \in I_P}$. Define $\Delta : C_P \to C_P \otimes C_P$ by lin. ext. of

$$\Delta(e_{[p,q]}) = \sum_{r \in [p,q]} e_{[p,r]} \otimes e_{[r,q]}$$

and $\varepsilon : C_P \to \mathbb{K}$ by lin. ext. of

$$\varepsilon(e_{[p,q]}) = \delta_{p,q} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}.$$

Then $(C_P, \Delta, \varepsilon)$ is a coalgebra. Checking (H1') and (H2') is very similar to the previous example.

Sweedler's sigma notation

Very soon when working with coalgebras, one starts needing a convenient and manageable notation for coproducts.

Let $C = (C, \Delta, \varepsilon)$ be a coalgebra. Recall that for any $x \in C$, the element $\Delta(x) \in C \otimes C$ in the second tensor power of C can be written as

$$\Delta(x) = \sum_{j=1}^{m} y_j \otimes z_j$$

for some $m \in \mathbb{N}$ and some elements $y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_m \in C$. But this expression is by no means unique!

Even the number m of terms in it is not the same for all possible expressions of this type (the minimal number of terms is called the rank of $\Delta(x)$, but it would actually be really inconvenient to always require such minimal expressions) and certainly e.g. the identities

$$\lambda y \otimes \lambda^{-1} z = y \otimes z \qquad (\lambda \in \mathbb{K})$$

$$y \otimes (z' + z'') = y \otimes z' + y \otimes z''$$

are further reasons for non-uniqueness.

For many calculations, however, it is important to use some expressions of this form.

It is also important to emphasize that the y_j's and the z_j's depend on the element x — although they are not uniquely determined.

Sweedler's sigma notation

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

stands for any such expression, and emphasizes the dependence of the RHS on x.

Now for example axiom (H2') reads

$$x = \sum_{(x)} \varepsilon(x_{(1)}) \cdot x_{(2)} = \sum_{(x)} x_{(1)} \cdot \varepsilon(x_{(2)})$$

and (H1') is the equality

$$\sum_{(x)} \sum_{(x_{(1)})} (x_{(1)})_{(1)} \otimes (x_{(1)})_{(2)} \otimes x_{(2)}$$
$$= \sum_{(x)} \sum_{(x_{(2)})} x_{(1)} \otimes (x_{(2)})_{(1)} \otimes (x_{(2)})_{(2)}$$

and we will use the expression $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ for these.

Let us take a second look at some of the bialgebra axioms.

In order to be algebra homomorphisms, the maps $\Delta : A \to A\otimes A$ and $\varepsilon : A \to \mathbb{K}$ must respect the products and units. These requirements for the map ε are

$$(H5') : \quad \varepsilon \circ \mu \cong \varepsilon \otimes \varepsilon \quad : \quad A\otimes A \to \mathbb{K}$$

$$\Big(\underbrace{\varepsilon(a * b)}_{= \varepsilon(\mu(a\otimes b))} = \varepsilon(a)\,\varepsilon(b) \qquad \forall a,b \in A \Big)$$

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & A \arrow[d, "\varepsilon"] \\ \mathbb{K}\otimes\mathbb{K} \arrow[r, leftrightarrow, "\cong"] & \mathbb{K} \end{tikzcd}$$

$$(H6) : \quad \varepsilon \circ \eta = \mathrm{id}_{\mathbb{K}} \quad : \quad \mathbb{K} \to \mathbb{K}$$

$$\big(\varepsilon(\eta(\lambda)) = \lambda \qquad \forall \lambda \in \mathbb{K} \big)$$

$$\begin{tikzcd} & A \arrow[dr, "\varepsilon"] & \\ \mathbb{K} \arrow[ur, "\eta"] \arrow[rr, equal] & & \mathbb{K} \end{tikzcd}$$

In order to formulate the requirements for Δ in a similar way, we use the tensor flip maps: for two vector spaces V and W, define $\tau_{V,W} : V\otimes W \to W \otimes V$ by linear extension of $\tau_{V,W}(v\otimes w) = w \otimes v$. The homomorphism property of Δ then amounts to:

$$(H4) : \quad \Delta \circ \mu = (\mu\otimes\mu)\circ(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)\circ(\Delta\otimes\Delta) : \quad A\otimes A \to A\otimes A$$

$$\begin{tikzcd} & A \arrow[dr, "\Delta"] & \\ A\otimes A \arrow[ur, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & & A\otimes A \\ A\otimes A\otimes A\otimes A \arrow[rr, "\mathrm{id}\otimes\tau_{A,A}\otimes\mathrm{id}"] & & A\otimes A\otimes A\otimes A \arrow[u, "\mu\otimes\mu"'] \end{tikzcd}$$

$$\boxed{\Delta(ab) = \sum_{(a)}\sum_{(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}}$$

(H5): $\Delta \circ \eta = \eta \otimes \eta \quad ; \quad \mathbb{K} \to A \otimes A$

$(\Delta(1_A) = 1_A \otimes 1_A)$

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A \otimes A \\ \mathbb{K} \arrow[u, "\eta"] \arrow[r, "\cong"] & \mathbb{K} \otimes \mathbb{K} \arrow[u, "\eta \otimes \eta"'] \end{tikzcd}$$

We may reformulate the definition of bialgebra as: a biolgebra is a quintuple $A = (A, \mu, \eta, \Delta, \varepsilon)$ where:

- A is a $\mathbb{K}$-vect. space
- $\mu : A \otimes A \to A$, $\eta : \mathbb{K} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{K}$ are linear maps that satisfy the axioms (H1), (H1'), (H2), (H2'), (H4), (H5), (H5'), (H6).

Of course, homomorphisms of coalgebras are defined as maps that respect the corresponding structural maps: the coproducts and counits.

Def: Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be two coalgebras. A linear map $f : C \to C'$ is a coalgebra homomorphism if $\Delta' \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

Remark: Properties (H4) and (H5') can thus be interpreted as saying that $\mu : A \otimes A \to A$ is a coalgebra homomorphism (equip $A \otimes A$ with the "tensor product coalgebra" structure) and properties (H5) and (H6) that $\eta : \mathbb{K} \to A$ is a coalgebra homomorphism (equip $\mathbb{K}$ with the coalgebra structure s.t. $\Delta_{\mathbb{K}}(\lambda) = \lambda \otimes 1$ and $\varepsilon_{\mathbb{K}}(\lambda) = \lambda$).

Of course there is also an isomorphism theorem for coalgebras. A few notions are needed for its formulation.

Def: Let (C, Δ, ε) be a coalgebra.
A vector subspace $C' \subset C$ is a *subcoalgebra* if $\Delta(C') \subset C' \otimes C'$.
A vector subspace $J \subset C$ is a *coideal* if $\Delta(J) \subset J \otimes C + C \otimes J$ and $\varepsilon|_J \equiv 0$.

Proposition If (C, Δ, ϵ) is a coalgebra and $J \subset C$ is a coideal, then the quotient vector space C/J becomes a coalgebra with the following structural maps:

coproduct : $c + J \longmapsto \sum_{(c)} (c_{(1)} + J) \otimes (c_{(2)} + J)$

counit : $c + J \longmapsto \varepsilon(c)$

(which are in particular well defined).

Theorem (Isomorphism theorem for coalgebras)
If (C, Δ, ε) and $(\tilde{C}, \tilde{\Delta}, \tilde{\varepsilon})$ are coalgebras and $f: C \to \tilde{C}$ is a homomorphism of coalgebras, then

1°) $\mathrm{Im}(f) \subset \tilde{C}$ is a subcoalgebra

2°) $\mathrm{Ker}(f) \subset C$ is a coideal

3°) $C/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ as coalgebras.

Hopf algebras

Bialgebras are enough for making sense of trivial representation and tensor products, but duals of representations and representations on spaces of linear maps more generally require yet some additional structure — namely a suitable map $\gamma : A \to A$ called the antipode.

Def: A Hopf algebra is a sextuple $A = (A, \mu, \eta, \Delta, \varepsilon, \gamma)$ where

- A is a vector space
- $\mu : A \otimes A \to A$, $\eta : \mathbb{K} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{K}$, $\gamma : A \to A$ are linear maps satisfying the axioms (H1), (H1'), (H2), (H2'), (H3) below, (H4), (H5), (H5'), (H6).

(H3): $\mu \circ (\gamma \otimes \mathrm{id}_A) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id}_A \otimes \gamma) \circ \Delta$

$A \to A$

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\gamma \otimes \mathrm{id}"] & & A \otimes A \arrow[d, "\mu"] \\
A \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\varepsilon"] & \mathbb{K} \arrow[r, "\eta"] & A \\
A \otimes A \arrow[rr, "\mathrm{id} \otimes \gamma"'] & & A \otimes A \arrow[u, "\mu"']
\end{tikzcd}$$

In Sweedler's notation:

$$\varepsilon(a) 1_A = \sum_{(a)} \gamma(a_{(1)})\, a_{(2)} = \sum_{(a)} a_{(1)}\, \gamma(a_{(2)})$$

Our two favourite examples provide the first examples of Hopf algebras as well.

<u>Example</u> (The group algebra is a Hopf algebra)

G group, $\mathbb{C}[G]$ its group algebra

basis $(u_g)_{g\in G}$, structural maps linear extensions of

$$\mu(u_g \otimes u_h) = u_{gh}$$

$$\eta(\lambda) = u_e \qquad (e \in G \text{ neutral element})$$

$$\Delta(u_g) = u_g \otimes u_g$$

$$\varepsilon(u_g) = 1$$

$$\gamma(u_g) = u_{g^{-1}} \qquad (g^{-1} \in G \text{ the inverse elem.})$$

We have already seen (more or less) that $\mathbb{C}[G]$ is a bialgebra, so it remains to check axiom (H3).

$$\mu \circ (\gamma \otimes \mathrm{id}) \circ \Delta \overset{?}{=} \eta \circ \varepsilon \overset{?}{=} \mu \circ (\mathrm{id} \otimes \gamma) \circ \Delta$$

LHS : $u_g \overset{\Delta}{\longmapsto} u_g \otimes u_g \overset{\gamma \otimes \mathrm{id}}{\longmapsto} u_{g^{-1}} \otimes u_g$

$\overset{\mu}{\longmapsto} u_{g^{-1}g} = u_e = 1_{\mathbb{C}[G]} = \varepsilon(u_g)\cdot \eta(1).$

so indeed the left equality holds. The other is similar.

<u>Example</u> (Polynomial algebra is a Hopf algebra)

$\mathbb{C}[t]$ = polynomial algebra in one indeterminate t.

basis $(t^n)_{n\in\mathbb{N}}$ of monomials, structural maps

$$\mu(t^n \otimes t^m) = t^{n+m}$$

$$\eta(\lambda) = \lambda \cdot t^0 \qquad (\text{unit is const. poly } 1)$$

$$\Delta(t^n) = \sum_{k=0}^{n} \binom{n}{k} t^k \otimes t^{n-k}$$

$$\varepsilon(t^n) = \delta_{n,0} = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

$$\gamma(t^n) = (-1)^n \cdot t^n$$

Again we have essentially checked before that $\mathbb{C}[t]$ is a bialgebra, so it remains to check (H3).

$$t^n \xrightarrow{\Delta} \sum_{k=0}^{n} \binom{n}{k} t^k \otimes t^{n-k} \xrightarrow{\gamma \otimes id} \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, t^k \otimes t^{n-k}$$

$$\xrightarrow{\mu} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot t^n = \left(1+(-1)\right)^n \cdot t^n$$

$$= \delta_{n,0} \cdot t^n = \varepsilon(t^n) \cdot t^0 = \varepsilon(t^n)\, 1_{\mathbb{C}[t]} .$$

So indeed $\mu \circ (\gamma \otimes id) \circ \Delta = \eta \circ \varepsilon$, and the other side of (H3) is verified similarly.

<u>Example</u> (Universal enveloping algebra of a Lie algebra)
For a Lie algebra $\mathfrak{g}$ there is an associative algebra $\mathcal{U}(\mathfrak{g})$, "the universal enveloping algebra", such that the representations of the Lie alg. $\mathfrak{g}$ correspond exactly to representations of the associative algebra $\mathcal{U}(\mathfrak{g})$ (compare: group G and group algebra $\mathbb{C}[G]$).
We have $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$ and the algebra $\mathcal{U}(\mathfrak{g})$ is generated by this subset $\mathfrak{g}$.
There is a unique Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ such that

$$\Delta(X) = X \otimes 1_{\mathcal{U}(\mathfrak{g})} + 1_{\mathcal{U}(\mathfrak{g})} \otimes X \qquad \forall\ X \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$$

which corresponds to the Leibniz rule, reflecting the fact that Lie algebra elements act as derivations. For this Hopf algebra structure we have in particular when $X \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$

$$\varepsilon(X) = 0 , \qquad \gamma(X) = -X .$$

Properties of the antipode

Recall that we denote by $S_{V,W} : V\otimes W \to W \otimes V$ the linear map switching the factors of the tensor product, i.e. the linear extension of $S_{V,W}(v\otimes w) = w \otimes v \quad \forall v \in V, w\in W$.

For an algebra $A = (A, \mu, \eta)$, we denote the opposite product by

$$\mu^{op} : A\otimes A \longrightarrow A \qquad\qquad \mu^{op} = \mu \circ S_{A,A}.$$

It is easy to see that $A^{op} = (A, \mu^{op}, \eta)$ is also an algebra. The algebra A is commutative if $\mu^{op} = \mu$.

For a coalgebra $C = (C, \Delta, \varepsilon)$, we denote the opposite coproduct by

$$\Delta^{op} : A \longrightarrow A\otimes A \qquad\qquad \Delta^{op} = S_{C,C} \circ \Delta.$$

It is easy to see that $C^{cop} = (C, \Delta^{op}, \varepsilon)$ is also a coalgebra. The coalgebra C is said to be cocommutative if $\Delta^{op} = \Delta$.

Exercise: Suppose that $B = (B, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra. Show that all of the following are also bialgebras:

$$B^{op} = (B, \mu^{op}, \eta, \Delta, \varepsilon) \qquad \text{"opposite bialgebra"}$$
$$B^{cop} = (B, \mu, \eta, \Delta^{op}, \varepsilon) \qquad \text{"co-opposite bialgebra"}$$
$$B^{op,cop} = (B, \mu^{op}, \eta, \Delta^{op}, \varepsilon) \qquad \text{"opposite co-opposite bialgebra"}$$

Exercise: Suppose that $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ is a Hopf algebra. Show that $H^{op,cop} = (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon, \gamma)$ is also a Hopf algebra. Show also that the following are equivalent:

1°) the bialgebra $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon)$ admits an antipode $\tilde{\gamma}$

2°) the bialgebra $H^{cop} = (H, \mu, \eta, \Delta^{op}, \varepsilon)$ admits an antipode $\tilde{\gamma}$

3°) γ is invertible with inverse $\tilde{\gamma}$.

The first important properties of the antipode that we want to establish is that it reverses products and coproducts. More precisely:

Theorem Let $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ be a Hopf algebra. Then we have

(i): $\gamma \circ \mu = \mu^{op} \circ (\gamma \otimes \gamma)$

(i.e. $\gamma(ab) = \gamma(b)\gamma(a) \quad \forall a, b \in H$)

(ii) $(\gamma \otimes \gamma) \circ \Delta = \Delta^{op} \circ \gamma$

(i.e. $\Delta(\gamma(a)) = \sum_{(a)} a_{(2)} \otimes a_{(1)} \quad \forall a \in H$)

Along the way, we end up also showing the uniqueness of the antipode (precise statement later).

A key tool for the proofs is convolution algebras.

Proposition/definition: Let $A = (A, \mu, \eta)$ be an algebra and $C = (C, \Delta, \varepsilon)$ a coalgebra. For any $f, g \in \mathrm{Hom}(C, A)$, define $f \star g \in \mathrm{Hom}(C, A)$ by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta : C \to A.$$

Then $\mathrm{Hom}(C, A)$ becomes an algebra with the product $\star$ and unit $1_\star = \eta \circ \varepsilon : C \to A$.

(This algebra is called the convolution algebra of C and A.)

Using Sweedler's notation for coproducts in C, we get an "elementwise" formula for $f \star g$:

$$(f \star g)(c) = \sum_{(c)} f(c_{(1)}) g(c_{(2)}) .$$

Proof of proposition/definition: Let us check unitality.

For $c \in C$ calculate:

$$(f \star 1_\star)(c) = \sum_{(c)} f(c_{(1)}) \cdot 1_\star(c_{(2)}) = \sum_{(c)} f(c_{(1)}) (\eta \circ \varepsilon)(c_{(2)})$$

$$= \sum_{(c)} f(c_{(1)})\, \varepsilon(c_{(2)})\, 1_A = \sum_{(c)} f(c_{(1)} \cdot \varepsilon(c_{(2)}))$$

$$\overset{(H2')}{=} f(c) \quad \Rightarrow \quad f \star 1_\star = f.$$

The property $1_\star \star g = g$ is similar.
For associativity, use coassociativity of C and associativity of A: if $f, g, h \in \mathrm{Hom}(C, A)$, then

$$((f \star g) \star h)(c) = \sum_{(c)} (f \star g)(c_{(1)}) \cdot h(c_{(2)})$$
$$= \sum_{(c)} \Big(\sum_{(c_{(1)})} f((c_{(1)})_{(1)}) \cdot g((c_{(1)})_{(2)}) \Big) h(c_{(2)})$$
$$\overset{(H1),(H1')}{=} \sum_{(c)} \sum_{(c_{(2)})} f(c_{(1)}) \Big(g((c_{(2)})_{(1)}) \, h((c_{(2)})_{(2)}) \Big)$$
$$= (f \star (g \star h))(c). \qquad \square$$

As the first application, let us prove the uniqueness of the antipode.

Lemma: Let $B = (B, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then if $\gamma : B \to B$ and $\tilde{\gamma} : B \to B$ are two linear maps satisfying (H3), they must coincide: $\gamma = \tilde{\gamma}$.

Proof: Form the convolution algebra $\mathrm{Hom}(B, B)$ (use the coalgebra structure and the algebra structure separately on the domain and range, respectively).
Consider the element $id_B \in \mathrm{Hom}(B, B)$.
Using property (H3) for γ, we compute

$$(\gamma \star id_B)(x) = \sum_{(x)} \gamma(x_{(1)}) x_{(2)} = \varepsilon(x) 1_B = (\eta \circ \varepsilon)(x)$$

$$\Longrightarrow \quad \gamma \star id_B = 1_\star .$$

Similarly using (H3) for $\tilde{\gamma}$ one gets $id_B \star \tilde{\gamma} = 1_\star$.
But the left and right inverses must coincide:

$$\gamma = \gamma \star 1_\star = \gamma \star (id_B \star \tilde{\gamma}) = (\gamma \star id_B) \star \tilde{\gamma}$$
$$= 1_\star \star \tilde{\gamma} = \tilde{\gamma}. \qquad \square$$

Along similar lines we may prove that it is enough to verify bialgebra homomorphism properties for a Hopf algebra homomorphism.

<u>Lemma</u>: Suppose that $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ and $H' = (H', \mu', \eta', \Delta', \varepsilon', \gamma')$ are two Hopf algebras, and suppose that $f : H \longrightarrow H$ is a bialgebra homomorphism, i.e. $\mu' \circ (f \otimes f) = f \circ \mu$, $\eta' = f \circ \eta$, $\Delta' \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$. Then f is in fact also a Hopf algebra homomorphism, i.e. $\gamma' \circ f = f \circ \gamma$ holds also.

<u>Proof</u>: Consider the convolution algebra $\mathrm{Hom}(H, H')$. A left inverse of f in $\mathrm{Hom}(H,H')$ is found by the following calculation:

$$\begin{aligned}
((f \circ \gamma) \star f)(x) &= \sum_{(x)} f(\gamma(x_{(1)})) \cdot f(x_{(2)}) \\
&\overset{\text{alg. homom.}}{=} \sum_{(x)} f(\gamma(x_{(1)})\, x_{(2)}) \overset{(H3)}{=} f(\varepsilon(x)\, 1_H) \\
&= \varepsilon(x)\, f(1_H) \overset{\text{alg. hom.}}{=} \varepsilon(x)\, 1_{H'} = (\eta' \circ \varepsilon)(x) = 1_\star(x).
\end{aligned}$$

A right inverse of f is found by

$$\begin{aligned}
(f \star (\gamma' \circ f))(x) &= \sum_{(x)} f(x_{(1)})\, \gamma'(f(x_{(2)})) \\
&\overset{\text{coalg. homom.}}{=} \sum_{(f(x))} (f(x))_{(1)}\, \gamma'((f(x))_{(2)}) \\
&\overset{(H3) \text{ for } \gamma'}{=} \varepsilon'(f(x)) \cdot 1_{H'} \overset{\text{coalg. hom.}}{=} \varepsilon(x)\, 1_{H'} = 1_\star(x).
\end{aligned}$$

The left and right inverses must coincide:

$$\begin{aligned}
f \circ \gamma &= (f \circ \gamma) \star 1_\star = (f \circ \gamma) \star (f \star (\gamma' \circ f)) \\
&= ((f \circ \gamma) \star f) \star (\gamma' \circ f) = 1_\star \star (\gamma' \circ f) = \gamma' \circ f.
\end{aligned}$$

$\square$

The proofs that the antipode reverses products and coproducts use similar ideas.

Proof of part (i) of Theorem:

We will in fact show that $S : H \to H^{op}$ is an algebra homomorphism, i.e., that it respects the products

$$\mu^{op} \circ (S \otimes S) = S \circ \mu$$

and units

$$\eta = S \circ \eta .$$

Let us first check the latter condition. Recall that 1_H is grouplike $(\Delta(1_H) = 1_H \otimes 1_H)$ so its counit is $\varepsilon(1_H) = 1$. Therefore (H3) gives

$$1_H = \varepsilon(1_H)\, 1_H = (\eta \circ \varepsilon)(1_H)$$

$$\overset{(H3)}{=} (\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta)(1_H) = S(1_H)\, 1_H = S(1_H).$$

It thus only remains to check the former condition $\mu^{op} \circ (S \otimes S) = S \circ \mu$. For this purpose, consider the convolution algebra $\mathrm{Hom}(H \otimes H, H)$, where the coalgebra structure on $H \otimes H$ is given by coproduct

$$\Delta_2 = (\mathrm{id}_H \otimes S_{H,H} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta)$$

i.e. $\Delta_2(x \otimes y) = \sum_{(x)} \sum_{(y)} x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$

and counit

$$\varepsilon_2 = \varepsilon \otimes \varepsilon \qquad \text{i.e.} \quad \varepsilon_2(x \otimes y) = \varepsilon(x)\varepsilon(y).$$

We claim that the element $\mu \in \mathrm{Hom}(H \otimes H, H)$ has left inverse $\mu^{op} \circ (S \otimes S)$ and right inverse $S \circ \mu$. The assertion then follows, because the left and right inverses must coincide.

Calculate first

$$(\mu \star (\mathcal{S}\circ\mu))(x\otimes y) = \sum_{(x\otimes y)} \mu((x\otimes y)_{(1)})\, \mathcal{S}(\mu((x\otimes y)_{(2)}))$$

$$= \sum_{(x)}\sum_{(y)} \mu(x_{(1)}\otimes y_{(1)})\, \mathcal{S}(\mu(x_{(2)}\otimes y_{(2)}))$$

$$= \sum_{(x)}\sum_{(y)} x_{(1)}y_{(1)}\, \mathcal{S}(x_{(2)}y_{(2)})$$

$$\overset{(H4)}{=} \sum_{(xy)} (xy)_{(1)}\, \mathcal{S}((xy)_{(2)}) \overset{(H3)}{=} \varepsilon(xy)\, 1_H$$

$$\overset{(H5')}{=} \varepsilon(x)\,\varepsilon(y)\, 1_H = \varepsilon_2(x\otimes y)\, 1_H = 1_\star(x\otimes y)$$

$$\Longrightarrow \quad \mu \star (\mathcal{S}\circ\mu) = 1_\star .$$

Then calculate

$$((\mu^{op}\circ(\mathcal{S}\otimes\mathcal{S}))\star\mu)(x\otimes y) = \sum_{(x\otimes y)} \cdots$$

$$= \sum_{(x)}\sum_{(y)} \mu^{op}(\mathcal{S}(x_{(1)})\otimes\mathcal{S}(y_{(1)}))\, \mu(x_{(2)}\otimes y_{(2)})$$

$$= \sum_{(x)}\sum_{(y)} \mathcal{S}(y_{(1)})\,\mathcal{S}(x_{(1)})\, x_{(2)}\, y_{(2)}$$

$$\overset{(H3)}{=} \sum_{(y)} \mathcal{S}(y_{(1)})\,\varepsilon(x)\, 1_H\, y_{(2)} \overset{(H3)}{=} \varepsilon(x)\cdot\varepsilon(y)\, 1_H$$

$$= \varepsilon_2(x\otimes y)\, 1_H = 1_\star(x\otimes y)$$

$$\Longrightarrow \quad (\mu^{op}\circ(\mathcal{S}\otimes\mathcal{S}))\star\mu = 1_\star .$$

This finishes the proof. $\square$

Proof of part (ii) of Theorem: We leave the details to the reader: the key idea is to check that in the convolution algebra $\mathrm{Hom}(H, H\otimes H)$ the left and right inverses of $\Delta : H \to H\otimes H$ are given by two expressions that we seek to prove equal. $\square$

Dual representations and representations on linear maps

Recall that with the coproduct $\Delta : H \to H \otimes H$ we can define tensor product representations, and with the counit $\varepsilon : H \to \mathbb{K}$ we can define the trivial representation. Among the most important remaining operations that we would like to be able to perform on representations is to equip the space of linear maps

$$\mathrm{Hom}(V, W) = \{ T : V \to W \text{ linear} \},$$

between two representations V and W, with the structure of a representation. Taking $W = \mathbb{K}$ the trivial representation, a special case of this is the dual space

$$V^* = \mathrm{Hom}(V, \mathbb{K}).$$

The antipode γ is exactly what is needed to achieve this!

<u>Proposition</u> Suppose $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ is a Hopf algebra and $\rho_V : H \to \mathrm{End}(V)$, $\rho_W : H \to \mathrm{End}(W)$ are two representations of H. Then the space $\mathrm{Hom}(V, W)$ becomes a representation of H by setting

$$x.T = \sum_{(x)} \rho_W(x_{(1)}) \circ T \circ \rho_V(\gamma(x_{(2)}))$$

for $x \in H$, $T \in \mathrm{Hom}(V, W)$.

<u>Proof</u>: Use the fact that $\gamma : H \to H^{op}$ is an algebra homomorphism. First of all, the unit $1_H \in H$ acts as (recall $\Delta(1_H) = 1_H \otimes 1_H$)

$$1_H . T = \underbrace{\rho_W(1_H)}_{= \mathrm{id}_W} \circ T \circ \underbrace{\rho_V(\underbrace{\gamma(1_H)}_{= 1_H})}_{= \mathrm{id}_V} = T$$

as it should.

The product of $x, y \in H$ acts as

$$(xy).T = \sum_{(xy)} \rho_W((xy)_{(1)}) \circ T \circ \rho_V(\gamma((xy)_{(2)}))$$

$$= \sum_{(x)} \sum_{(y)} \rho_W(x_{(1)} y_{(1)}) \circ T \circ \rho_V(\underbrace{\gamma(x_{(2)} y_{(2)})}_{= \gamma(y_{(2)}) \gamma(x_{(2)})})$$

$$= \sum_{(x)} \sum_{(y)} \rho_W(x_{(1)}) \circ \rho_W(y_{(1)}) \circ T \circ \rho_V(\gamma(y_{(2)})) \circ \rho_V(\gamma(x_{(2)}))$$

$$= \sum_{(x)} \rho_W(x_{(1)}) \circ (y.T) \circ \rho_V(\gamma(x_{(2)}))$$

$$= x.(y.T).$$ $\square$

So the fact that $\gamma : H \to H^{op}$ is an algebra homomorphism was enough to equip $\mathrm{Hom}(V,W)$ and in particular $V^* = \mathrm{Hom}(V, \mathbb{K})$ with the structure of a representation (compare with algebra homomorphism properties of $\varepsilon : H \to \mathbb{K}$ and $\Delta : H \to H \otimes H$). But does this representation have natural properties? (As the trivial representation and tensor product representations did).

For an algebra A and two representations $\rho_V : A \to \mathrm{End}(V)$, $\rho_W : A \to \mathrm{End}(W)$, the space of <u>A-intertwining maps</u> (or <u>A-module maps</u>) is

$$\mathrm{Hom}_A(V,W) = \{ T : V \to W \text{ linear} \mid \rho_W(a) \circ T = T \circ \rho_V(a) \quad \forall a \in A,\ v \in V \}$$

$$\subset \mathrm{Hom}(V,W).$$

For a bialgebra B and a representation $\rho_U : B \to \mathrm{End}(U)$ the subspace of invariants is

$$U^B = \{ u \in U \mid \rho_U(b).u = \varepsilon(b)\, u \quad \forall b \in B \ \forall u \in U \}.$$

In other words, U^B is the trivial subrepresentation in U: it consists of all vectors $u \in U$ s.t. the 1-dimensional space $\mathbb{K}u \subset U$ is a trivial representation.

For Hopf algebras we have:

<u>Theorem</u> Let $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ be a Hopf algebra and $\rho_V : H \to \mathrm{End}(V)$, $\rho_W : H \to \mathrm{End}(W)$ two representations of H. Then we have

$$\underbrace{\mathrm{Hom}_H(V,W)}_{\substack{\text{space of}\\ \text{intertwining maps}\\ \text{from } V \text{ to } W}} = \underbrace{\mathrm{Hom}(V,W)^H}_{\substack{\text{the trivial subrepresentation}\\ \text{in the representation}\\ \text{on linear maps } V \to W.}}.$$

<u>Proof:</u> "$\subset$": Assume $T \in \mathrm{Hom}_H(V,W)$. For $x \in H$ and $v \in V$, calculate:

$$(x.T)(v) = \sum_{(x)} x_{(1)}.\, T(\gamma(x_{(2)}).v)$$

$$\overset{\text{intertwining}}{=} T\Big(\sum_{(x)} x_{(1)}\cdot \gamma(x_{(2)}).v\Big)$$

$$\overset{(H3)}{=} T(\varepsilon(x)\, 1_H.v) = \varepsilon(x)\, T(v)$$

$$\Longrightarrow \quad x.T = \varepsilon(x)\, T$$

"$\supset$": Assume $T \in \mathrm{Hom}(V,W)^H$, i.e. $\forall x \in H$

$$x.T = \varepsilon(x)\, T.$$

Now write $x = \sum_{(x)} x_{(1)}\, \varepsilon(x_{(2)})$ by (H2'). and calculate for any $v \in V$

$$x.(T(v)) = \sum_{(x)} x_{(1)}\, \varepsilon(x_{(2)}).(T(v))$$

$$= \sum_{(x)} x_{(1)}.\big(T(\varepsilon(x_{(2)})\, 1_H.v)\big) \qquad \begin{pmatrix}\text{linearity and}\\ 1_H.v = v\end{pmatrix}$$

$$\overset{(H3)}{=} \sum_{(x)} x_{(1)}.\Big(T\Big(\sum_{(x_{(2)})} \gamma((x_{(2)})_{(1)})(x_{(2)})_{(2)}.v\Big)\Big)$$

$$\overset{(H1')}{=} \sum_{(x)} \sum_{(x_{(1)})} (x_{(1)})_{(1)}.\big(T(\gamma((x_{(1)})_{(2)})\, x_{(2)}.v\big)$$

$$= \sum_{(x)} (x_{(1)}.T)(x_{(2)}.v) = \sum_{(x)} \varepsilon(x_{(1)})\, T(x_{(2)}.v)$$

$$= T\Big(\sum_{(x)} \varepsilon(x_{(1)})\, x_{(2)}.v\Big) \overset{(H2')}{=} T(x.v).$$

$\square$

On the order of the antipode

Let us finish by some remarks on the iterations of the antipode, $\gamma \circ \gamma \circ \cdots \circ \gamma : H \to H$.

<u>Proposition</u>: If a Hopf algebra $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ is either commutative ($\mu^{op} = \mu$) or cocommutative ($\Delta^{op} = \Delta$), then the antipode is involutive: $\gamma \circ \gamma = \mathrm{id}_H$.

<u>Proof</u>: Suppose H is cocommutative, i.e. $\Delta = \Delta^{op} =: S_{H,H} \circ \Delta$. Consider the convolution algebra $\mathrm{Hom}(H,H)$, and recall (from the proof of uniqueness of antipode) that $\mathrm{id}_H \in \mathrm{Hom}(H,H)$ is the inverse of the element $\gamma \in \mathrm{Hom}(H,H)$. Now calculate using cocommutativity

$$\big(\gamma \star (\gamma \circ \gamma)\big)(x) = \sum_{(x)} \gamma(x_{(1)})\, \gamma(\gamma(x_{(2)}))$$

$$\overset{\gamma : H \to H^{op}}{=} \sum_{(x)} \gamma\big(\gamma(x_{(2)})\, x_{(1)}\big) \overset{\text{cocomm.}}{=} \sum_{(x)} \gamma\big(\gamma(x_{(1)})\, x_{(2)}\big)$$

$$\overset{(H3)}{=} \gamma(\varepsilon(x) 1_H) = \varepsilon(x) \underbrace{\gamma(1_H)}_{= 1_H} = \varepsilon(x) 1_H = 1_\star(x)$$

Therefore also $\gamma \circ \gamma$ is a right inverse of γ in the convolution algebra $\mathrm{Hom}(H,H)$. The right inverse $\gamma \circ \gamma$ must coincide with the left inverse id_H:

$$\mathrm{id}_H = \mathrm{id}_H \star 1_\star = \mathrm{id}_H \star \big(\gamma \star (\gamma \circ \gamma)\big)$$

$$= (\mathrm{id}_H \star \gamma) \star (\gamma \circ \gamma) = 1_\star \star (\gamma \circ \gamma) = \gamma \circ \gamma.$$

The commutative case is very similar. $\square$

It is not always true that the antipode is an involution. The next example shows that it may not even be of finite order.

<u>Exercise</u> Let $q \in \mathbb{C} \setminus \{0\}$, and let H_q be the algebra generated by elements a, a', b subject to relations

$$a a' = 1_{H_q} = a' a \quad , \qquad ab = q \cdot ba .$$

Because of the first relation, we have $a' = a^{-1}$ in H_q.

Show that there is a unique Hopf algebra structure on H_q such that

$$\Delta(a) = a \otimes a \quad , \qquad \Delta(b) = a \otimes b + b \otimes 1_{H_q} .$$

Show also that $\gamma(a) = a^{-1}$, $\gamma(b) = -a^{-1} b$ and conclude in particular that

$$\gamma(\gamma(b)) = \frac{1}{q} b .$$

If q is not a root of unity, $q^n \neq 1$ for all $n \in \mathbb{Z} \setminus \{0\}$ then the above Hopf algebra H_q has antipode γ whose iterates are all distinct, e.g.

$$(\underbrace{\gamma \circ \gamma \circ \cdots \circ \gamma}_{2n})(b) = q^{-n} \cdot b .$$

By the previous proposition, this behavior can only occur for non-commutative, non-cocommutative Hopf algebras.

Braided Hopf-Algebras:

Going from groups to group algebras:

$g \in G$ (group) $\xrightarrow[g \mapsto e_g]{\text{homomorphic injection}}$ $e_g \in \mathbb{C}[G]$ (group algebra)

$\rho_V : G \to \mathrm{Aut}(V)$ (representation) $\qquad \rho_V : \mathbb{C}[G] \to \mathrm{End}\, V$

---

1. $\rho_{trivial} : G \to \mathrm{Aut}(\mathbb{C}) = \mathbb{C}$ $\qquad\longrightarrow\qquad$ $\rho_{trivial} : \mathbb{C}[G] \to \mathrm{End}\,\mathbb{C} = \mathbb{C}$

   $\rho_{trivial}(g) = 1 \quad \forall g \in G$ $\qquad\qquad$ $\rho_{trivial}(a) = \varepsilon(a)$

2. $\rho_{V\otimes W} : G \to \mathrm{Aut}(V\otimes W)$ $\qquad\longrightarrow\qquad$ $\rho_{V\otimes W} : \mathbb{C}[G] \to \mathrm{End}(V\otimes W)$

   $\rho_{V\otimes W}(g) = (\rho_V \otimes \rho_W)(g\otimes g)$ $\qquad\qquad$ $\rho_{V\otimes W}(a) = [(\rho_V \otimes \rho_W)\circ \Delta](a)$

3. $\rho_{Hom(V,W)} : G \to \mathrm{Aut}(\mathrm{Hom}(V,W))$ $\qquad\longrightarrow\qquad$ $\rho_{Hom(V,W)} : \mathbb{C}[G] \to \mathrm{End}(\mathrm{Hom}(V,W))$

   $\rho_{Hom(V,W)}(g)(T) = \rho_W(g)\circ T \circ \rho_V(g^{-1})$ $\qquad$ $\rho_{Hom(V,W)}(a) = \sum_{(a)} \rho_W((a)_1) \circ T \circ \rho_V(\gamma((a)_2))$

where $\varepsilon, \Delta, \gamma$ are given by linear ext. of

$$\varepsilon(e_g) = 1, \quad \Delta(e_g) = e_g \otimes e_g, \quad \gamma(e_g) = e_{g^{-1}}$$

The coproduct and counit axioms give

$$V\otimes\mathbb{C} \cong V \cong \mathbb{C}\otimes V \quad \text{(as representations)}$$

for any rep. V. The coproduct axioms give

$$V_1\otimes(V_2\otimes V_3) \cong (V_1\otimes V_2)\otimes V_3 \quad \text{(as representations)}$$

for any rep.s V_1, V_2, V_3. The antipode and coproduct axioms give

$$W\otimes V^* \cong \mathrm{Hom}(V,W) \quad \text{(as representations)}$$

for any rep.s W, V. (Here, V^* becomes a rep. of $\mathbb{C}[G]$ by setting $W = \mathbb{C}$ in 3.).

Going from Lie algebras to universal enveloping algebras:

$A \in \mathfrak{g}$ (Lie algebra) $\xrightarrow[\substack{A \to e_A \ \{A\} \text{ basis} \\ [A,B] \to e_A e_B - e_B e_A}]{\text{homomorphic injection}}$ $e_A \in \mathcal{U}(\mathfrak{g})$ (universal enveloping algebra)

$\rho_V : A \to \mathrm{End}(V)$ (representation) $\qquad \rho_V : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}\, V$

---

1. $\rho_{trivial} : \mathfrak{g} \to \mathrm{End}\, \mathbb{C} = \mathbb{C}$ $\qquad \rho_{trivial} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}\, \mathbb{C} = \mathbb{C}$

   $\rho_{trivial}(A) = 0 \quad \forall A \in \mathfrak{g}$ $\qquad \rho_{trivial}(a) = \varepsilon(a)$

2. $\rho_{V\otimes W} : \mathfrak{g} \to \mathrm{End}\, V\otimes W$ $\qquad \rho_{V\otimes W} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V\otimes W)$

   $\rho_{V\otimes W}(A) = \rho_V(A) \otimes id_W + id_V \otimes \rho_W(A)$ $\qquad \rho_{V\otimes W}(a) = [(\rho_V \otimes \rho_W) \circ \Delta](a)$

3. $\rho_{Hom(V,W)} : \mathfrak{g} \to \mathrm{End}(Hom(V\otimes W))$ $\qquad \rho_{Hom(V,W)} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(Hom(V,W))$

   $\rho_{Hom(V,W)}(A)(T) = -T \circ \rho_V(A) - \rho_W(A) \circ T$ $\qquad \rho_{Hom(V,W)}(a) = \sum_{(a)} \rho_W((a)_1) \circ T \circ \rho_V(\gamma((a)_2))$

where $\varepsilon, \Delta, \gamma$ are given by linear ext. of

$$\varepsilon(e_A) = 0, \quad \Delta(e_A) = e_A \otimes 1_A + 1_A \otimes e_A, \quad \gamma(e_A) = e_{-A} = -e_A$$

"

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"

There is one more "natural" isomorphism of vector spaces,

$$S_{V,W} = S : V \otimes W \xrightarrow{S} W \otimes V \quad , \quad S(v \otimes w) = w \otimes v$$

that we would like to be an isomorphism of representatio[ns]

whenever V and W are reps. of $G, \mathbb{C}[G], \mathfrak{g}, U(\mathfrak{g})$.

That is, we want σ to be an intertwiner.

$$[S \circ \rho_{V\otimes W}(a)](v \otimes w) = [\rho_{W \otimes V}(a) \circ S](v \otimes w) \quad \forall v \in V, w \in W, a$$

This holds if Δ is "cocommutative", i.e.

$$\Delta = \Delta^{op} := S \circ \Delta$$

Indeed, if Δ is cocommutative, then

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} = \sum_{(a)} a_{(2)} \otimes a_{(1)} = \Delta^{op}(a), \text{ so}$$

$$\Rightarrow [S \circ \rho_{V\otimes W}(a)](v \otimes w) = S\Big(\sum_{(a)} \rho_V(a_{(2)})(v) \otimes \rho_W(a_{(1)})(w)\Big)$$

$$= \sum_{(a)} \rho_W(a_{(1)})(w) \otimes \rho_V(a_{(2)})(v)$$

$$= [\rho_{W\otimes V}(a)](w \otimes v) = [\rho_{W \otimes V}(a) \circ S](v \otimes w)$$

Because Δ is cocommutative for $\mathbb{C}[G]$ (resp. $U(\mathfrak{g})$),

it follows that $V \otimes W \cong W \otimes V$ for any two reps.

V, W of $\mathbb{C}[G]$ (resp. $U(\mathfrak{g})$), with S the isomorphism.

Representations of cocommutative Hopf algebras:

$\Delta = \Delta^{op}$

Theorem 1: Let H be a cocommutative Hopf algebra, and let V, W, V', W' be representations of H.

1. $S_{V,W} : V \otimes W \longrightarrow W \otimes V$, $S_{V,W}(v \otimes w) = w \otimes v$ linearly extended is an isomorphism of representations.

2. Let $f : V \to V'$ and $g : W \to W'$ be intertwiners. Then

$$S_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ S_{V,W}$$

3. The representation $c_n : \mathfrak{S}_n \longrightarrow \mathrm{Aut}(V^{\otimes n})$ of the symmetric group $\mathfrak{S}_n$ of order $n!$, given by homomorphic extension of

$$c_n(\sigma_i)(v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes v_{i+2} \otimes \cdots \otimes v_n)$$
$$= v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n$$

where σ_i = i^{th} transposition $(1 \leq i \leq n-1)$

generators of $\mathfrak{S}_n$

is well-defined, and

4. $$c_n(\sigma) \circ \rho_{V^{\otimes n}}(a) = \rho_{V^{\otimes n}}(a) \circ c_n(\sigma) \quad \forall\ \sigma \in \mathfrak{S}_n, a \in H$$

Proof:

1. $S_{V,W}$ is obviously a linear bijection. As we showed above it is also an intertwiner because $\Delta = \Delta^{op}$. ✓
2. This obviously holds for all such maps f, g.
3. With

$$\mathfrak{S}_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i ,\ |i-j| > 1 \\ \sigma_i^2 = id\ \forall\ i,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}\ \forall\ i < n \end{array} \right\rangle$$

we must check that the group relations are satisfied under c:

$$c(\sigma_i^2) \overset{?}{=} id_{V^{\otimes n}}, \quad c(\sigma_i \sigma_{i+1} \sigma_i) \overset{?}{=} B_n(\sigma_{i+1} \sigma_i \sigma_{i+1}).$$

These equalities are easy to verify using the homomorphism property of c.

4. $$\rho_{V^{\otimes n}}(a) = (\rho_V \otimes \rho_V \otimes \cdots \otimes \rho_V) \circ \Delta^{(n)}(a), \quad \text{where}$$

(exercise 2) $$\Delta^{(n)} := (\Delta \otimes id^{\otimes(n-2)}) \circ \Delta^{(n-1)}, \quad \Delta^{(2)} := \Delta$$

By coassociativity, we also have

$$\Delta^{(n)} = (\mathrm{id}^{\otimes(n-2)} \otimes \Delta) \circ \Delta^{(n-1)}$$

To prove that $c_n(\sigma)$ and $\rho_{V^{\otimes n}}(a)$ commute, we use the fact that, with $\Delta^{op} = \Delta$, we have

$$\Delta^{(n)}(a) := \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(i-1)} \otimes a_{(i)} \otimes a_{(i+1)} \otimes a_{(i+2)} \otimes \cdots \otimes$$

$*$

$$= \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(i-1)} \otimes a_{(i+1)} \otimes a_{(i)} \otimes a_{(i+2)} \otimes \cdots \otimes$$

$\forall\ a \in H$ and $\forall\ i \leq n-1$.

$$[\rho_{V^{\otimes n}}(a) \circ c(\sigma_i)](\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots) = \rho_{V^{\otimes n}}(a)(\cdots \otimes v_{i+1} \otimes v_i \otimes \cdots$$

$$= \sum_{(a)} \cdots \otimes \rho_V(a_{(i+1)}) \otimes \rho_V(a_{(i)}) \otimes \cdots \quad (\cdots \otimes v_{i+1} \otimes v_i \otimes$$

using the 2nd expression for $\Delta^{(n)}(a)$ $= \sum_{(a)} \cdots \otimes \rho_V(a_{(i+1)})(v_{i+1}) \otimes \rho_V(a_{(i)})(v_i) \otimes \cdots$

using the 1st expression for $\Delta^{(n)}(a)$ $= c(\sigma_i) \sum_{(a)} \cdots \otimes \rho_V(a_{(i)})(v_i) \otimes \rho_V(a_{(i+1)})(v_{i+1}) \otimes \cdots$

$$= c(\sigma_i) \circ \rho_{V^{\otimes n}}(a)$$

What remains is to prove $*$. That is, if

$$S_i = \mathrm{id}^{\otimes(i-1)} \otimes H \otimes H \otimes \mathrm{id}^{\otimes(n-i-1)} : H^{\otimes n} \to H^{\otimes n}$$

switches the i^{th} and $(i+1)^{th}$ tensorands of $a \in H^{\otimes n}$, then $S_i \circ \Delta^{(n)} = \Delta^{(n)} \quad \forall\ i < n$.

Proof by induction: This is cocommutativity for $n=2$.

Assume $S_i \circ \Delta^{(n-1)} = \Delta^{(n-1)}\ \forall\ i < n-1$. Then

$$S_i \circ \Delta^{(n)} = S_i \circ (\mathrm{id}^{\otimes(n-2)} \circ \Delta) \circ \Delta^{(n-1)} = (\mathrm{id}^{\otimes(n-2)} \circ \Delta) \circ S_i \circ$$

$$= (\mathrm{id}^{\otimes(n-2)} \circ \Delta) \circ \Delta^{(n-1)} = \Delta^{(n)} \quad \text{for } i < n-1.$$

$$S_{n-1} \circ \Delta^{(n)} = S_{n-1} \circ (\Delta \otimes \mathrm{id}^{\otimes(n-2)}) \circ \Delta^{(n-1)}$$

$$= (\mathrm{id}^{\otimes(n-2)} \otimes \Delta) \circ S_{n-1} \circ \Delta^{(n-1)} = (\mathrm{id}^{\otimes(n-2)} \otimes \Delta) \circ \Delta$$

$$= \Delta^{(n)}.$$

If H is not coassociative (i.e. $\Delta \neq \Delta^{op}$), then $S_{V,W}$ is typically not an isomorphism of H-representations.

Representations of braided Hopf algebras:

Q: When can we find a map $c_{V,W} : V \otimes W \to W \otimes V$ that satisfies (almost) all properties of $S_{V,W}$ in theorem 1?

A: How should $c_{V,W}$ differ from $S_{V,W}$? Well, it would probably not satisfy $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$. For $V = W$, this condition $S_{V,V}^2 = \mathrm{id}_{V \otimes V}$ corresponds to the property $\sigma_i^2 = \mathrm{id}_{V^{\otimes n}}$ for G_n in item 3 of the theorem. If we drop this relation from G_n, then we have

Def: The braid group with n strands:

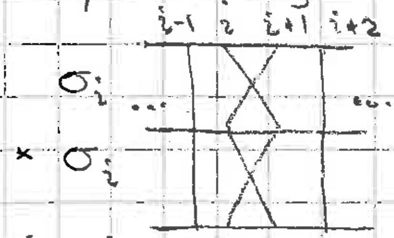
$$Br_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall \ i < n-1 \end{array} \right\rangle$$

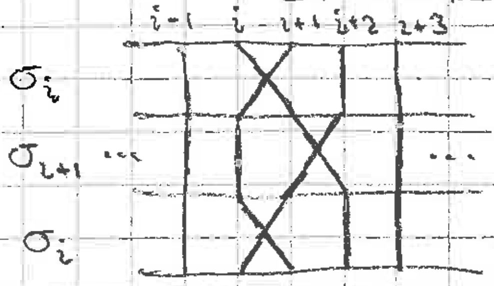
We find a nice diagram rep. of Br_n from the following diagram rep. of G_n:

$\mathrm{id} =$ 

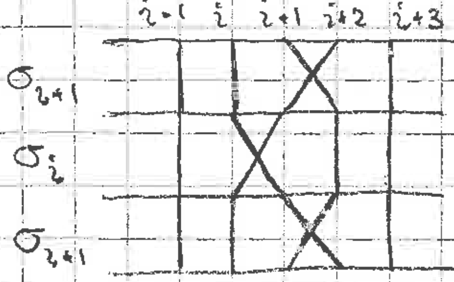
$\sigma_i = \sigma_i^{-1} =$ 

Multiply by stacking vertically:

$\sigma_i^2 = \sigma_i \times \sigma_i$  $=$  $= \mathrm{id}$

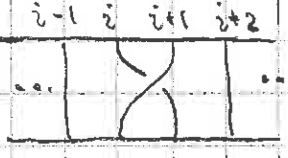

$\sigma_i \sigma_{i+1} \sigma_i$

$=$


$\sigma_{i+1} \sigma_i \sigma_{i+1}$

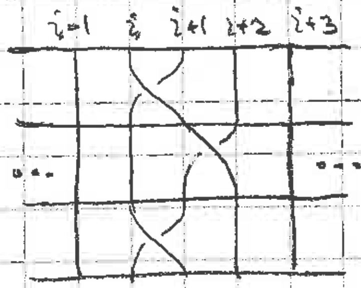
For $\sigma_i \in Br_n$, we do not have $\sigma_i = \sigma_i^{-1}$. To distinguish them, we write

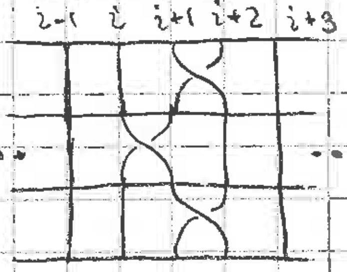
$\sigma_i =$ 

, $\sigma_i^{-1} =$ 

$\in Br_n$

Then

σ_i
σ_{i+1}
σ_i


$=$
σ_{i+1}
σ_i
σ_{i+1}


$\sigma_i \sigma_{i+1} \sigma_i$ $\qquad\qquad$ $\sigma_{i+1} \sigma_i \sigma_{i+1}$

(One should check that the diagrams themselves don't satisfy extra relations too.) To reformulate Thm 1 in terms of $(c_{V,W}, Br_n)$ replacing $(S_{V,W}, \mathfrak{S}_n)$, we need

c1. $c_{V,W} : V \otimes W \to W \otimes V$ to be as iso. of H-reps.

c2. $c_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W}$ for any two intertwiners $f : V \to V'$, $g : W \to W'$

c3. $c : Br_n \to \mathrm{Aut}(V^{\otimes n})$ given by homomorphic extension of

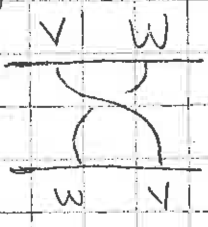
$$c(\sigma_i)(\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots) = \cdots \otimes c_{V,V}(v_i \otimes v_{i+1}) \otimes \cdots$$

is well-defined i.e., for $c_i := c(\sigma_i)$,

$$c_i \circ c_{i+1} \circ c_i = c_{i+1} \circ c_i \circ c_{i+1}, \quad i < n-1$$

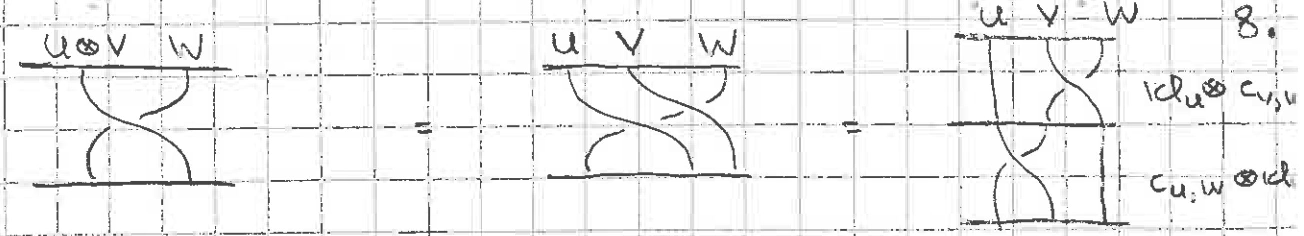
c4. and $c(\sigma) \circ \rho_{V^{\otimes n}}(a) = \rho_{V^{\otimes n}}(a) \circ c(\sigma)$

We write the diagrams

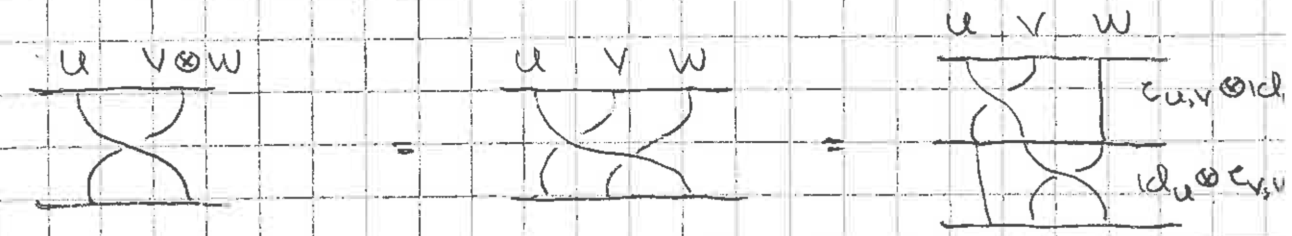
$c_{V,W} =$ 

$\qquad c_{V,W}^{-1} =$ 

Now we consider the "natural" conditions



c5. $c_{u\otimes v, w} = (c_{u,w} \otimes id_v) \circ (id_u \otimes c_{v,w})$



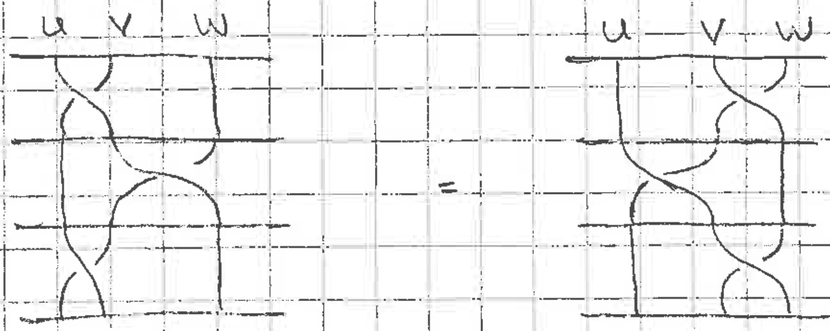
c6. $c_{u, v\otimes w} = (id_v \otimes c_{u,w}) \circ (c_{u,v} \otimes id_w)$

Theorem 2: If $c_{v,w} : V \otimes W \to W \otimes V$ satisfies c1, c2, c5, c6, then

$$(c_{v,w} \otimes id_u) \circ (id_v \otimes c_{u,w}) \circ (c_{u,v} \otimes id_w) \quad \text{(Yang-Baxter eqn.)}$$

c7. $$= (id_w \otimes c_{u,v}) \circ (c_{u,w} \otimes id_v) \circ (id_u \otimes c_{v,w})$$

That is,



Proof: $(c_{v,w} \otimes id_u) \circ (id_v \otimes c_{u,w}) \circ (c_{u,v} \otimes id_w)$

$\underset{c6}{=} (c_{v,w} \otimes id_u) \circ c_{u, v\otimes w}$

$= c_{u, w\otimes v} \circ (id_u \otimes c_{v,w})$ by c2. with $f = id_u$, $g = c_{v,w}$

$\underset{c5}{=} (id_w \otimes c_{u,v}) \circ (c_{u,w} \otimes id_v) \circ (id_u \otimes c_{v,w})$ ✓

Note: If $U = V = W$ and $c_1 = c_{v,v} \otimes id_v$, $c_2 = id_v \otimes c_{v,v}$, then we have $c_1 \circ c_2 \circ c_1 = c_2 \circ c_1 \circ c_2$, which is c3 for the case $n = 3$. $\Rightarrow$ $c_{v,v}$ satisfies c3 too.

Thus, if $c_{V,W}$ satisfies c1, c2, c5, c6 $\forall$ H-rep. V, W, then $c_{V,V}$ satisfies c3 too. As it will turn out, such a $c_{V,V}$ will satisfy c4 as well. Thus, we look for a "universal" way to construct such a family $c_{V,W}$. For any V, W, writing

$$c_{V,W} = \underbrace{S_{V,W}}_{\text{"switch"}} \circ \underbrace{R_{V,W}}_{\text{"R-matrix"}}, \quad R_{V,W} : V \otimes W \to V \otimes W \text{ linear}$$

We posit the existence of an $R \in H \otimes H$ s.t.

$$R_{V,W} = (\rho_V \otimes \rho_W)(R), \text{ i.e., } c_{V,W} = S_{V,W} \circ (\rho_V \otimes \rho_W)(R).$$

What would R have to satisfy so $c_{V,W}$ satisfies c1, c2, c5, c6?

automatic, from onset for $c_{V,W}$

c1. $\rightsquigarrow$ R is invertible in $H \otimes H$ (R1)

c1. $\rightsquigarrow$ $R\Delta(a) = \Delta^{op}(a) R \quad \forall\, a \in H$ (R2)

Next, let $R = \sum_i s_i \otimes t_i \implies R_{13} := \sum_i s_i \otimes 1_H \otimes t_i$

$\implies R_{12} := \sum_i s_i \otimes t_i \otimes 1_H, \quad R_{23} := \sum_i 1_H \otimes s_i \otimes t_i$

$(\Delta \otimes id)(R)$ 1 2 3

c5. $\rightsquigarrow$  (R3)

$$(\Delta \otimes id)(R) = R_{13} R_{23}$$

$(id \otimes \Delta)(R)$ 1 2 3

c6. $\rightsquigarrow$  (R4)

$$(id \otimes \Delta)(R) = R_{13} R_{12}$$

Def: An $R \in H \otimes H$ satisfying R1-R4 is called a "universal R-matrix." If $H \otimes H$ has a universal R-matrix, then we say that H is "braided."

Ex. If H is cocommutative, then it is braided with

$$R = 1_H \otimes 1_H \Rightarrow c_{V,W} = S_{V,W} \circ (\rho_V \otimes \rho_W)(R) = S_{V,W}.$$

Ex. $H = \mathbb{C}[\mathbb{Z}_N]$, the group (Hopf) algebra of the cyclic group is braided with at least two universal R-matrices:

$$R_1 = 1_H \otimes 1_H \quad \text{(b.c. } H \text{ is cocommutative)}$$

$$R = \frac{1}{N} \sum_{j,k=0}^{N-1} e^{2\pi i jk/N}\, \Theta^j \otimes \Theta^k, \quad \Theta = \text{generator of } \mathbb{Z}_N.$$

Thus, a universal R-matrix is not unique

Ex. Let H be the Hopf algebra generated by two element x, y with $x^2 = 1$, $y^2 = 0$, $xy + yx = 0$. Then $\{1_H, x, y, xy\}$ is a basis for H, and we take

$$\Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1, \qquad \gamma(x) = x$$

$$\Delta(y) = 1_H \otimes y + y \otimes x, \qquad \varepsilon(y) = 0, \qquad \gamma(y) = xy.$$

Then $\forall\ \lambda \in \mathbb{C}$,

$$R_\lambda = \tfrac{1}{2}(1_H \otimes 1_H + 1_H \otimes x + x \otimes 1_H - x \otimes x)$$
$$+ \tfrac{\lambda}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y)$$

is a universal R-matrix of $H \otimes H$, so H is braided

Theorem 3: If H is braided with universal R-matrix $R \in H \otimes H$ then

$$c_{V,W} := S_{V,W} \circ (\rho_V \otimes \rho_W)(R)$$

satisfies c1 - c4. (This theorem is analogous to Theorem 1)

Proof:

c1. R1 $\Rightarrow$ R is invertible. Then $c_{V,W}$ is invertible (thus injec

with $(c_{V,W})^{-1} = (S_{V,W} \circ \rho_V \otimes \rho_W(R))^{-1}$

$$= (\rho_V \otimes \rho_W(R))^{-1} \circ S_{V,W}^{-1}$$

$$= (\rho_V \otimes \rho_W(R^{-1})) \circ S_{W,V} \qquad R^{-1}\ \exists \text{ by c1}$$

Thus $c_{V,W} : V \otimes W \to W \otimes V$ is a v. space iso.

Next, we prove it is an H-rep. hom. (i.e. intertwiner).

$\forall\ a \in H,\ v \in V,\ w \in W$

$$\begin{aligned}
c_{V,W}(a.(v\otimes w)) &= \left[S_{V,W}\circ(\rho_V\otimes\rho_W)(R)\right](a.(v\otimes w)) \\
&= \left[S_{V,W}\circ(\rho_V\otimes\rho_W)(R)\right]\circ\left[\rho_V\otimes\rho_W(\Delta(a))\right](v\otimes w) \\
&= \left[S_{V,W}\circ(\rho_V\otimes\rho_W)(R\,\Delta(a))\right](v\otimes w) \\
&\overset{(R2)}{=} \left[S_{V,W}\circ(\rho_V\otimes\rho_W)(\Delta^{op}(a)R)\right](v\otimes w) \\
&= \left[(\rho_W\otimes\rho_V)(\Delta(a))\right]\circ\left[S_{V,W}\circ(\rho_V\otimes\rho_W)(R)\right](v\otimes w) \\
&= a.\left[S_{V,W}\circ(\rho_V\otimes\rho_W)(R)\right](v\otimes w) \\
&= a.\ c_{V,W}(v\otimes w)\ \checkmark
\end{aligned}$$

c2. This is immediate from the fact that f and g are H-rep. hom.: With $R = \sum_i s_i\otimes t_i$,

$$\begin{aligned}
c_{V',W'}\circ(f\otimes g) &= \sum_i S_{V',W'}\circ\rho_V(s_i)\otimes\rho_W(t_i)\circ(f\otimes g) \\
&= \sum_i S_{V',W'}\circ(f\otimes g)\circ\rho_V(s_i)\otimes\rho_W(t_i) \\
&= \sum_i (g\otimes f)\circ S_{V,W}\circ\rho_V(s_i)\otimes\rho_W(t_i) \\
&= (g\otimes f)\circ c_{V,W}
\end{aligned}$$

c3. By theorem 2, it suffices to show that c5, c6 hold:

$$\begin{aligned}
c_{U\otimes V,W} &= S_{U\otimes V,W}\circ\rho_{U\otimes V}\otimes\rho_W(R), \quad R = \sum_i s_i\otimes t_i \\
&= S_{U\otimes V,W}\circ(\rho_U\otimes\rho_V\otimes\rho_W)(\Delta\otimes id)(R) \\
&\overset{(R3)}{=} S_{U\otimes V,W}\circ(\rho_U\otimes\rho_V\otimes\rho_W)(R_{13}R_{23}) \\
&= S_{U\otimes V,W}\circ(\rho_U\otimes\rho_V\otimes\rho_W)(R_{13})\circ(\rho_U\otimes\rho_V\otimes\rho_W)(R_{23}) \\
&= (S_{U,W}\otimes id_V)\circ(id_U\otimes S_{V,W})\circ\sum_i\rho_U(s_i)\otimes id_V\otimes\rho_W(t_i) \\
&\quad\circ\sum_j id_U\otimes\rho_V(s_j)\otimes\rho_W(t_j) \\
&= (S_{U,W}\otimes id_V)\circ\sum_i\rho_U(s_i)\otimes\rho_W(t_i)\otimes id_V\circ(id_U\otimes S_{V,W}) \\
&\quad\circ\sum_j id_U\otimes\rho_V(s_j)\otimes\rho_W(t_j) \\
&= (S_{U,W}\circ\rho_U\otimes\rho_W(R)\otimes id_V)\circ(id_U\otimes S_{V,W}\circ\rho_V\otimes\rho_W(R)) \\
&= (c_{U,W}\otimes id_V)\circ(id_U\otimes c_{V,W})\ \checkmark
\end{aligned}$$

A similar proof gives c6 from R4. Thus, c3 holds.

c4. We have $\forall\, a \in H$, $i < n$

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$$c(\sigma_i)\circ \rho_{V^{\otimes n}}(a) = \left(id_V^{\otimes(i-1)} \otimes c_{V,V} \otimes id_V^{\otimes(n-i-1)}\right)\circ(\rho_V\otimes\cdots\otimes\rho_V)(\Delta^{(n)}(a))$$

Now write $\Delta^{(n)}(a) = \left(id^{\otimes(i-1)}\otimes\Delta\otimes id^{\otimes(n-i-1)}\right)\circ\Delta^{(n-1)}(a)$

$$= \sum_{(a)} a_{(1)}\otimes a_{(2)}\otimes\cdots\otimes a_{(i-1)}\otimes\Delta(a_i)\otimes a_{(i+2)}\otimes\cdots$$

$$\Rightarrow c(\sigma_i)\circ\rho_{V^{\otimes n}}(a) = \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes c_{V,V}\circ(\rho_V\otimes\rho_V)(\Delta(a))\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes S_{V,V}\circ(\rho_V\otimes\rho_V)(R)\circ(\rho_V\otimes\rho_V(\Delta(a)))\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes S_{V,V}\circ(\rho_V\otimes\rho_V)(R\,\Delta(a))\circ\rho_V(a_{(i+2)})\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes S_{V,V}\circ(\rho_V\otimes\rho_V)(\Delta^{op}(a)R)\circ\rho_V(a_{(i+2)})\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes S_{V,V}\circ(\rho_V\otimes\rho_V)(\Delta^{op}(a))\circ\rho_V\otimes\rho_V(R)\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes(\rho_V\otimes\rho_V)(\Delta(a))\circ S_{V,V}\circ\rho_V\otimes\rho_V(R)\otimes\cdots$$

$$= \sum_{(a)}\cdots\otimes\rho_V(a_{(i-1)})\otimes(\rho_V\otimes\rho_V)(\Delta(a))\circ c_{V,V}\otimes\rho_V(a_{(i+2)})\otimes\cdots$$

$$= \rho_{V^{\otimes n}}(a)\circ c(\sigma_i).$$

Ex. Let $q\in\mathbb{C}^\times\setminus\{\pm1\}$, $p := \min\{n\in\mathbb{Z}^+ \mid q^n = \pm1\} < \infty$. Let $\bar{U}_q$ be the Hopf algebra w. generators $\{E, F, K^{\pm1}\}$ and relations

$$KE = q^2EK,\quad KF = q^{-2}FK,\quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

$$E^p = F^p = 0,\quad K^p = 1.$$

The coproduct, counit, and antipode are

$$\Delta(E) = 1\otimes E + E\otimes K,\quad \Delta(F) = K^{-1}\otimes F + F\otimes 1,\quad \Delta(K) = K\otimes K$$

$$\varepsilon(E) = \varepsilon(F) = 0,\quad \varepsilon(K) = 1$$

$$\gamma(E) = -EK^{-1},\quad \gamma(F) = -KF,\quad \gamma(K) = K^{-1}$$

A basis for $\bar{U}_q$ is $\{E^iF^jK^\ell\}_{0\le i,j,\ell<p}$.

$\bar{U}_q$ is braided w. universal R-matrix

$$R = \frac{1}{p} \sum_{i,j,k=0}^{p-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k K^i \otimes F^k K^j$$

Let $V = \operatorname{span}\{v_0, v_1\}$, $\dim V = 2$ be a rep. of $\bar{U}_q$ w.

$$E.v_0 = 0, \quad F.v_0 = v_1, \quad K.v_0 = q v_0$$
$$E.v_1 = v_0, \quad F.v_1 = 0, \quad K.v_1 = q^{-1} v_1$$

Then we can calculate

$$c_{V,V}(v_0 \otimes v_0) = \lambda q \, v_0 \otimes v_0$$
$$c_{V,V}(v_0 \otimes v_1) = \lambda \, v_1 \otimes v_0$$
$$c_{V,V}(v_1 \otimes v_0) = \lambda (v_0 \otimes v_1 + (q-q^{-1}) v_1 \otimes v_0)$$
$$c_{V,V}(v_1 \otimes v_1) = \lambda q \, v_1 \otimes v_1$$

$$\lambda := q^{(p-1)/2}$$

and $c: Br_n \to \operatorname{Aut}(V^{\otimes n})$

$$c(\sigma_i) = \mathrm{id}_V^{\otimes(i-1)} \otimes c_{V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)} : V^{\otimes n} \to V^{\otimes n}$$

is a rep. of Br_n.

Note: Nowhere in here did we use the antipode. By dropping γ, we arrive w. braided bi-algebras.

TAKING DUALS OF HOPF ALGEBRAS ETC

Linear algebra : duals and transposes

Let us start with the basics. For a $\mathbb{K}$-vector space V, the dual is

$$V^* = \mathrm{Hom}(V, \mathbb{K}) = \{\varphi : V \to \mathbb{K} \text{ linear map}\}.$$

We usually denote the evaluation of $\varphi \in V^*$ on a vector $v \in V$ by $\langle \varphi, v \rangle := \varphi(v)$. The evaluation is of course bilinear, and thus defines

$$\mathrm{ev}_V : V^* \otimes V \longrightarrow \mathbb{K}, \quad (\varphi, v) \longmapsto \langle \varphi, v \rangle.$$

This obvious map will have a non-trivial role later in our construction of knot invariants.

Taking the dual is a (contravariant) functor : if V and W are $\mathbb{K}$-vector spaces and $f : V \to W$ is a linear map, then we get a map $f^* : W^* \to V^*$, called the transpose of f, by

$$f^*(\varphi) = \varphi \circ f \in V^* \qquad \text{for any } \varphi \in W^*$$

$$\text{i.e.} \qquad \langle f^*(\varphi), v \rangle = \langle \varphi, f(v) \rangle \qquad \forall \varphi \in W^*, v \in V.$$

Moreover, if $f : V \to W$, $g : W \to U$ are linear maps, then $g \circ f : V \to U$ is linear and $(g \circ f)^* = f^* \circ g^*$ because for any $\varphi \in U^*$ we have

$$(g \circ f)^*(\varphi) = \varphi \circ (g \circ f) = (\varphi \circ g) \circ f$$

$$= g^*(\varphi) \circ f = f^*(g^*(\varphi)).$$

(This reversal of the order of composition is why the functor is said to be contravariant.)

Recall also that if V_1, V_2, W_1, W_2 are vector spaces and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2$ linear maps, then we have used

$$f_1 \otimes f_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

defined by linear extension of

$$v_1 \otimes v_2 \longmapsto f_1(v_1) \otimes f_2(v_2) \in W_1 \otimes W_2$$

for all $v_1 \in V_1$, $v_2 \in V_2$.

The association of $f_1 \otimes f_2$ to the pair (f_1, f_2) is clearly itself bilinear, so we get a linear mapping (which is injective (THINK CAREFULLY!))

$$\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \hookrightarrow \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2).$$

In particular, setting $W_1 = \mathbb{K}$, $W_2 = \mathbb{K}$ and identifying $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, we get an injective linear mapping (embedding)

$$V_1^* \otimes V_2^* \hookrightarrow \underbrace{(V_1 \otimes V_2)^*}_{\mathrm{Hom}(V_1 \otimes V_2, \mathbb{K}) \cong \mathrm{Hom}(V_1 \otimes V_2, \mathbb{K} \otimes \mathbb{K})}.$$

Concretely, to $\varphi_1 \in V_1^*$, $\varphi_2 \in V_2^*$, this embedding associates the dual element $\varphi_1 \otimes \varphi_2$ of $V_1 \otimes V_2$ given by

$$\left\langle \varphi_1 \otimes \varphi_2 , \sum_{s=1}^{r} v_1^{(s)} \otimes v_2^{(s)} \right\rangle = \sum_{s=1}^{r} \varphi_1(v_1^{(s)}) \cdot \varphi_2(v_2^{(s)}).$$

With this embedding, we interpret

$$V_1^* \otimes V_2^* \subset (V_1 \otimes V_2)^*,$$

but note that if V_1 and V_2 are infinite dimensional, the inclusion is strict (not equality). If V_1, V_2, W_1, W_2 are all finite dimensional, then of course the dimensions of both $\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2)$ and $\mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)$ are $\dim(V_1)\dim(V_2)\dim(W_1)\dim(W_2)$, so the injective map must also be bijective.

Let us record one important property of the embedding in relation to transposes.

<u>Lemma</u> Let V_1, V_2, W_1, W_2 be vector spaces and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2$ linear maps. Then the linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ has transpose $(f_1 \otimes f_2)^*$ which makes the following diagram commute:

$$\begin{tikzcd}
(V_1 \otimes V_2)^* & (W_1 \otimes W_2)^* \arrow[l, "(f_1 \otimes f_2)^*"'] \\
V_1^* \otimes V_2^* \arrow[u, hook, "\iota_V"'] & W_1^* \otimes W_2^* \arrow[l, "f_1^* \otimes f_2^*"] \arrow[u, hook, "\iota_W"']
\end{tikzcd}$$

.

<u>Proof</u>: Let $\varphi_1 \in W_1^*$, $\varphi_2 \in W_2^*$ and consider the two results in $(V_1 \otimes V_2)^*$ that we get by applying the two compositions on the vector $\varphi_1 \otimes \varphi_2 \in W_1^* \otimes W_2^*$. The two linear maps are determined by their values on such simple tensors, so this is sufficient to show their equality. The result of either of the composition is an element in the dual $(V_1 \otimes V_2)^*$ of $V_1 \otimes V_2$, and it is determined by its values on simple tensors of the form $v_1 \otimes v_2 \in V_1 \otimes V_2$.

Consider first the composition $\iota_V \circ (f_1^* \otimes f_2^*)$:

$$\begin{aligned}
&\langle (\iota_V \circ (f_1^* \otimes f_2^*))(\varphi_1 \otimes \varphi_2), v_1 \otimes v_2 \rangle \\
&= \langle \iota_V(f_1^*(\varphi_1) \otimes f_2^*(\varphi_2)), v_1 \otimes v_2 \rangle \\
&= \langle f_1^*(\varphi_1), v_1 \rangle \cdot \langle f_2^*(\varphi_2), v_2 \rangle = \langle \varphi_1, f_1(v_1) \rangle \cdot \langle \varphi_2, f_2(v_2) \rangle .
\end{aligned}$$

Consider then the composition $(f_1 \otimes f_2)^* \circ \iota_W$:

$$\begin{aligned}
\langle (f_1 \otimes f_2)^*(\iota_W(\varphi_1 \otimes \varphi_2)), v_1 \otimes v_2 \rangle &= \langle \iota_W(\varphi_1 \otimes \varphi_2), (f_1 \otimes f_2)(v_1 \otimes v_2) \rangle \\
&= \langle \iota_W(\varphi_1 \otimes \varphi_2), f_1(v_1) \otimes f_2(v_2) \rangle = \langle \varphi_1, f_1(v_1) \rangle \cdot \langle \varphi_2, f_2(v_2) \rangle .
\end{aligned}$$

$\square$

The dual of a coalgebra

Let (C, Δ, ε) be a coalgebra. The linear maps

$$\Delta : C \to C \otimes C \qquad \text{satisfy (H1'), (H2')}.$$
$$\varepsilon : C \to \mathbb{K}$$

The adjoints are

$$\Delta^* : (C \otimes C)^* \longrightarrow C^*$$
$$\varepsilon^* : \mathbb{K}^* \longrightarrow C^*.$$

We may identify $\mathbb{K}^* \cong \mathbb{K}$ (a functional $\varphi \in \mathbb{K}^*$ is identified with its value $\varphi(1) \in \mathbb{K}$ at $1 \in \mathbb{K}$).
Moreover, since $C^* \otimes C^* \subset (C \otimes C)^*$, we may restrict Δ^* to a map $C^* \otimes C^* \longrightarrow C^*$.
With structural maps thus constructed, the dual C^* of the coalgebra becomes an algebra.

Theorem Let $C = (C, \Delta, \varepsilon)$ be a coalgebra.
Define $A = C^*$, $\mu : A \otimes A \to A$ by $\mu = \Delta^*|_{C^* \otimes C^*}$, and $\eta : \mathbb{K} \to A$ by $\eta = \varepsilon^*$ with the identification $\mathbb{K}^* \cong \mathbb{K}$.
Then (A, μ, η) is an algebra.

Proof: By the above discussion $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are well defined, so we must only verify associativity (H1) and unitality (H2).
Start with unitality: let $f \in A = C^*$ and $1_A := \eta(1) = \varepsilon \in C^*$ and calculate, evaluating $\mu(1_A \otimes f) = \Delta^*(\varepsilon \otimes f)$ on any $c \in C$:

$$\langle \mu(1_A \otimes f), c \rangle = \langle \Delta^*(\varepsilon \otimes f), c \rangle = \langle \varepsilon \otimes f, \Delta(c) \rangle$$
$$= \sum_{(c)} \langle \varepsilon \otimes f, c_{(1)} \otimes c_{(2)} \rangle = \sum_{(c)} \varepsilon(c_{(1)}) \langle f, c_{(2)} \rangle$$
$$= \langle f, \sum_{(c)} \varepsilon(c_{(1)}) c_{(2)} \rangle \overset{(H2')}{=} \langle f, c \rangle,$$

which implies $\mu(1_A \otimes f) = f$ as desired.
Similarly $\mu(f \otimes 1_A) = f$, so (H2) follows.

For associativity, suppose $f, g, h \in A = C^*$ and calculate, evaluating on $c \in C$:

$$\langle \mu(f \otimes \mu(g \otimes h)), c \rangle = \langle f \otimes \mu(g \otimes h), \Delta(c) \rangle$$

$$= \sum_{(c)} \langle f, c_{(1)} \rangle \underbrace{\langle \mu(g \otimes h), c_{(2)} \rangle}_{= \langle g \otimes h, \Delta(c_{(2)}) \rangle}$$

$$= \sum_{(c)} \sum_{(c_{(2)})} \langle f, c_{(1)} \rangle \langle g, (c_{(2)})_{(1)} \rangle \langle h, (c_{(2)})_{(2)} \rangle$$

$$\overset{(H1')}{=} \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle \langle h, c_{(3)} \rangle .$$

Similarly $\langle \mu(\mu(f \otimes g) \otimes h), c \rangle = \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle \langle h, c_{(3)} \rangle$
so indeed $\mu(\mu(f \otimes g) \otimes h) = \mu(f \otimes \mu(g \otimes h))$. $\square$

A more visual approach is to apply transposes to the commutative diagrams (H1') and (H2').
For example (H1') says

$$\begin{tikzcd} & C \otimes C \arrow[dr, "\Delta \otimes id"] & \\ C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & C \otimes C \otimes C \\ & C \otimes C \arrow[ur, "id \otimes \Delta"'] & \end{tikzcd}$$

Using here the earlier lemma $(f_1 \otimes f_2)^*|_{W_1^* \otimes W_2^*} = f_1^* \otimes f_2^*$:

$$\begin{tikzcd} & (C \otimes C)^* \supset C^* \otimes C^* \arrow[dl, "\Delta^*"'] & (C \otimes C)^* \otimes C^* \arrow[l, "\Delta^* \otimes id_{C^*}"'] \\ C^* & & C^* \otimes C^* \otimes C^* \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\cap"] \\ & (C \otimes C)^* \supset C^* \otimes C^* \arrow[ul, "\Delta^*"] & C^* \otimes (C \otimes C)^* \arrow[l, "id_{C^*} \otimes \Delta^*"] \end{tikzcd}$$

and functoriality of duals, we get the commutative diagram (H1).

The restricted dual of an algebra

Since the dual of a coalgebra is an algebra — essentially by reversing the arrows of the structural maps and axioms — it seems natural to expect that the dual of an algebra would be a coalgebra. But there is one issue! For $A = (A, \mu, \eta)$ an algebra, the transpose of the product $\mu : A \otimes A \to A$ is

$$\mu^* : A^* \longrightarrow (A \otimes A)^* \supset A^* \otimes A^*,$$

which takes values in a space $(A \otimes A)^*$ that is a priori larger than the desired range $A^* \otimes A^*$ of a coproduct for A^*.

A solution is to look at only a suitable subspace in the dual A^* of an algebra, namely the restricted dual

$$A^\circ = (\mu^*)^{-1}(A^* \otimes A^*)$$

defined precisely as the preimage of the desired range of the coproduct.

We will show that

- the restricted dual of an algebra is a coalgebra
- the restricted dual of a Hopf algebra is a Hopf algebra.

This requires some preparations, however — in particular we must show that $\mu^*(A^\circ)$ is contained in $A^\circ \otimes A^\circ \subset A^* \otimes A^* \subset (A \otimes A)^*$.

Example (Representative forms)

Let $A = (A, \mu, \eta)$ be an algebra and $\rho : A \to \mathrm{End}(V)$ a representation of A on a finite-dimensional vector space V.
(Denote $\rho(a)v = a.v$ for $a \in A$, $v \in V$.)
Let $u_1, u_2, \ldots, u_n$ be a basis of V.
Then we can write, for all $a \in A$,

$$\rho(a) u_j = \sum_{i=1}^{n} \langle \lambda_{ij}, a \rangle u_i \qquad (\forall j = 1, \ldots, n)$$

where we have used the linear dependence of the result on $a \in A$ to write the coefficients in terms of some dual elements $\lambda_{ij} \in A^*$.
The dual elements λ_{ij} $(i,j = 1, \ldots, n)$ are called the representative forms of the representation V in the basis $u_1, u_2, \ldots, u_n$.
Now for $a, b \in A$ we have $\rho(ab) = \rho(a) \circ \rho(b)$
so

$$\rho(ab) u_j = \rho(a)\, \rho(b)\, u_j = \rho(a) \sum_{k=1}^{n} \langle \lambda_{kj}, b \rangle u_k$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \langle \lambda_{ik}, a \rangle \langle \lambda_{kj}, b \rangle u_i ,$$

and since this is also $\sum_{i=1}^{n} \langle \lambda_{ij}, ab \rangle u_i$
by definition, we get

$$\langle \lambda_{ij}, ab \rangle = \sum_{k=1}^{n} \langle \lambda_{ik}, a \rangle \langle \lambda_{kj}, b \rangle .$$

But now we observe that

$$\langle \mu^*(\lambda_{ij}), a \otimes b \rangle = \langle \lambda_{ij}, \mu(a \otimes b) \rangle = \langle \lambda_{ij}, ab \rangle$$

and

$$\langle \lambda_{ik} \otimes \lambda_{kj}, a \otimes b \rangle = \langle \lambda_{ik}, a \rangle \cdot \langle \lambda_{kj}, b \rangle ,$$

so the representative forms λ_{ij} are in the restricted dual:

$$\mu^*(\lambda_{ij}) = \sum_{k=1}^{n} \lambda_{ik} \otimes \lambda_{kj} \in A^* \otimes A^* .$$

Our first goal is to show that the restricted dual A° consists precisely of the representative forms of finite-dimensional representations of A.

We give a characterization of the restricted dual A° in terms of the left action of A on the dual A^* by

$$\langle a.f , b\rangle := \langle f, ba\rangle \qquad \text{for } f\in A^*,\ a\in A$$

and the right action

$$\langle f.a , b\rangle := \langle f, ab\rangle$$

<u>NOTE</u>: The right action makes A^* a representation of the opposite algebra A^{op} via $\rho : A^{op} \to \mathrm{End}(A^*)$ given by $\rho(a)f = f.a$.

<u>Lemma</u> The following are equivalent:

(i) $f\in A^\circ \subset A^*$

(ii) $A.f \subset A^*$ is a finite-dimensional subrepr.

(iii) $f.A \subset A^*$ is a finite-dim. subrepresentation (of the rep. A^* of the opposite alg. A^{op}).

<u>Proof</u> "(i) $\Rightarrow$ (ii)": Suppose $f\in A^\circ$, i.e. $\mu^*(f) = \sum_{i=1}^n g_i\otimes h_i$ for some $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n \in A^*$.

Then for $a\in A$, $x\in A$

$$\begin{aligned}\langle a.f , x\rangle &= \langle f, xa\rangle = \langle f, \mu(x\otimes a)\rangle \\ &= \langle \mu^*(f), x\otimes a\rangle = \sum_{i=1}^n \langle g_i\otimes h_i , x\otimes a\rangle \\ &= \sum_{i=1}^n \langle g_i, x\rangle\langle h_i, a\rangle = \Big\langle \sum_{i=1}^n \langle h_i, a\rangle g_i , x\Big\rangle .\end{aligned}$$

This shows $a.f = \sum_{i=1}^n \langle h_i, a\rangle g_i$, so $a.f$ always lies in the linear span of $g_1, g_2, \ldots, g_n$, so $A.f$ is finite-dim.

"(ii) $\Rightarrow$ (i)": Suppose $A.f \subset A^*$ is finite-dimensional, and choose a basis $g_1, \ldots, g_r$ of $A.f$. Then we can write

$$a.f = \sum_{i=1}^{r} \langle h_i, a \rangle g_i$$

for some $h_1, \ldots, h_r \in A^*$ by linearity of the action of A. Now for any $a, b \in A$

$$\langle \mu^*(f), a \otimes b \rangle = \langle f, \mu(a \otimes b) \rangle = \langle f, ab \rangle$$
$$= \langle b.f, a \rangle = \sum_{i=1}^{r} \langle h_i, b \rangle \langle g_i, a \rangle$$
$$= \langle \sum_{i=1}^{r} g_i \otimes h_i \, , \, a \otimes b \rangle$$

so we have $\mu^*(f) = \sum_{i=1}^{r} g_i \otimes h_i \in A^* \otimes A^*$ and thus $f \in A^\circ$.

"(i) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (i)" are similar. $\square$

<u>Remark</u>: It follows from the above proof that the rank of the tensor $\mu^*(f) \subset A^* \otimes A^*$ (for $f \in A^\circ$) is equal to the dimension $\dim(A.f)$ of the suprepresentation $A.f \subset A^*$. In fact, when

$$\mu^*(f) = \sum_{i=1}^{r} g_i \otimes h_i \quad \text{with } r \text{ minimal},$$

then $(g_i)_{i=1}^{r}$ is a basis of $A.f \subset A^*$ and $(h_i)_{i=1}^{r}$ is a basis of $f.A \subset A^*$.

From the above lemma we draw the first important conclusion towards making A° a coalgebra, namely $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$.

<u>Corollary</u> If $f \in A^\circ$, then we have

$$\mu^*(f) \in (A.f) \otimes (f.A) \subset A^\circ \otimes A^\circ.$$

<u>Proof</u> Above we wrote $\mu^*(f) = \sum_{i=1}^{r} g_i \otimes h_i$ with $g_i \in A.f$ and $h_i \in f.A$, so the first part follows. But clearly $A.f \subset A^\circ$, since for any $a \in A$ also $A.(a.f) \subset A.f$ is finite-dimensional. $\square$

We can also conclude (details left to reader):

Corollary The restricted dual A° of an algebra A is spanned by the representative forms of finite-dimensional representations of A.

With these preparations we get:

Theorem Let $A = (A, \mu, \eta)$ be an algebra. Define the space $C = A^\circ$ and structural maps $\Delta = \mu^*|_{A^\circ}$ and $\varepsilon = \eta^*|_{A^\circ}$. Then (C, Δ, ε) is a coalgebra. (Possibly $C = \{0\}$, though)

Sketch of proof: The corollaries above show that $\Delta : C \to C \otimes C$ is well defined, and obviously $\varepsilon : C \to \mathbb{K}$ is. It remains to check coassociativity (H1') and counitality (H2'). These are obtained by taking transposes of the maps in the commutative diagrams (H1), (H2) for A, and restricting to the appropriate subspaces.

For example, for counitality, let $f \in C = A^\circ$. Note that $\eta^*(f) \in \mathbb{K}^*$ is interpreted as the scalar $\langle \eta^*(f), 1_{\mathbb{K}} \rangle = \langle f, \eta(1_{\mathbb{K}}) \rangle = \langle f, 1_A \rangle$ so $\varepsilon(f) = \langle f, 1_A \rangle$. Write

$$\Delta(f) = \mu^*(f) = \sum_{i=1}^{r} g_i \otimes h_i .$$

Then calculate for $x \in A$

$$\langle (\varepsilon \otimes \mathrm{id}_C)(\Delta(f)), x \rangle = \sum_{i=1}^{r} \varepsilon(g_i) \langle h_i, x \rangle$$

$$= \sum_{i=1}^{r} \langle g_i, 1_A \rangle \langle h_i, x \rangle = \sum_{i=1}^{r} \langle g_i \otimes h_i, 1_A \otimes x \rangle$$

$$= \langle \mu^*(f), 1_A \otimes x \rangle = \langle f, \mu(1_A \otimes x) \rangle = \langle f, 1_A \rangle$$

so $((\varepsilon \otimes \mathrm{id}) \circ \Delta)(f) = f$. The rest is similar. $\square$

Also the restricted dual of a Hopf algebra is a Hopf algebra. For this, one needs two more lemmas.

Lemma Let $B = (B, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then we have $\Delta^*(B^\circ \otimes B^\circ) \subset B^\circ$ and $\mu^*(\varepsilon^*(1)) = \varepsilon^*(1) \otimes \varepsilon^*(1)$ so that $\varepsilon^*(1) \in B^\circ$.

Proof: See Lemma 3.51 in the typeset notes. $\square$
(or do as an exercise)

Lemma Let $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ be a Hopf alg. Then we have $\gamma(H^\circ) \subset H^\circ$.

Proof See Lemma 3.52 in the typeset notes. $\square$
(or do as an exercise)

Theorem Let $H = (H, \mu, \eta, \Delta, \varepsilon, \gamma)$ be a Hopf alg. On the space $H^\circ = (\mu^*)^{-1}(H^* \otimes H^*)$ define structural maps

$$\mu_{H^\circ} = \Delta^*|_{H^\circ \otimes H^\circ} \qquad \Delta_{H^\circ} = \mu^*|_{H^\circ}$$

$$\eta_{H^\circ} = \varepsilon^* \qquad \varepsilon_{H^\circ} = \eta^*|_{H^\circ}$$

$$\gamma_{H^\circ} = \gamma^*|_{H^\circ}.$$

Then $(H^\circ, \mu_{H^\circ}, \eta_{H^\circ}, \Delta_{H^\circ}, \varepsilon_{H^\circ}, \gamma_{H^\circ})$ is a Hopf alg.

Sketch of proof: By above lemmas, the structural maps map between the appropriate spaces. Now take transposes of the nine axioms for H and carefully keep track of suitable subspaces... $\square$

DRINFELD DOUBLE

Suppose that $A = (A, \mu, \eta, \Delta, \varepsilon, \gamma)$ is a Hopf algebra whose antipode $\gamma : A \to A$ is invertible (inverse $\gamma^{-1} : A \to A$). Let $B \subset A^\circ$ be a Hopf subalgebra of the restricted dual of A.

Recall: the structural maps of A° are

product: $\Delta^* |_{A^\circ \otimes A^\circ}$

unit: ε^*

coproduct: $\mu^* |_{A^\circ}$

counit: $\eta^* |_{A^\circ}$

antipode: $\gamma^* |_{A^\circ}$

which are the transposes of the structural maps of A, restricted to the appropriate subspaces.

Notation: Denote by 1^* the unit of A° (and therefore also of B), and recall:

$$\langle 1^*, a \rangle = \varepsilon(a) \qquad \forall a \in A .$$

Denote the coproduct of $\varphi \in A^\circ$ by

$$\mu^*(\varphi) = \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)}$$

so that for any $a, b \in A$ we have

$$\langle \varphi, ab \rangle = \langle \varphi, \mu(a \otimes b) \rangle = \langle \mu^*(\varphi), a \otimes b \rangle$$

$$= \Big\langle \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)}, a \otimes b \Big\rangle = \sum_{(\varphi)} \langle \varphi_{(1)}, a \rangle \cdot \langle \varphi_{(2)}, b \rangle .$$

Theorem (Drinfeld double)

Let A and $B \subset A^\circ$ be as above. Then the space $\mathcal{D}(A,B) = A \otimes B$ admits a unique Hopf algebra structure such that

(i) the map $\iota_A : A \to \mathcal{D}(A,B)$ given by $a \mapsto a \otimes 1^* \in A \otimes B$ is a Hopf algebra homomorphism

(ii) the map $\iota_B : B^{cop} \to \mathcal{D}(A,B)$ given by $\varphi \mapsto 1 \otimes \varphi \in A \otimes B$ is a Hopf algebra homomorphism

(iii) for all $a \in A$, $\varphi \in B$ we have
$$(a \otimes 1^*)(1 \otimes \varphi) = a \otimes \varphi$$

(iv) for all $a \in A$, $\varphi \in B$ we have
$$(1 \otimes \varphi)(a \otimes 1^*)$$
$$= \sum_{(a)} \sum_{(\varphi)} \langle \varphi_{(1)}, a_{(3)} \rangle \langle \varphi_{(3)}, \gamma^{-1}(a_{(1)}) \rangle \; a_{(2)} \otimes \varphi_{(2)}$$

Proof of uniqueness claim: Let us first check that the conditions (i), (ii), (iii), (iv) uniquely specify the structural maps of $\mathcal{D}(A,B)$, which we denote by

$$\mu_{\mathcal{D}}, \; \eta_{\mathcal{D}}, \; \Delta_{\mathcal{D}}, \; \varepsilon_{\mathcal{D}}, \; \gamma_{\mathcal{D}}.$$

Assuming $\mu_{\mathcal{D}}$ is associative and (iii), (iv) hold, for any $a, b \in A$, $\varphi, \psi \in B$, we have

$$(a \otimes \varphi)(b \otimes \psi) = (a \otimes 1^*)\underbrace{(1 \otimes \varphi)\,(b \otimes 1^*)}(1 \otimes \psi)$$

$$= \sum_{(b),(\varphi)} \langle \varphi_{(1)}, b_{(3)} \rangle \langle \varphi_{(3)}, \gamma^{-1}(b_{(1)}) \rangle \; \underbrace{b_{(2)} \otimes \varphi_{(2)}}_{= (b_{(2)} \otimes 1^*)(1 \otimes \varphi_{(2)})}$$

$$= \sum_{(b),(\varphi)} \langle \varphi_{(1)}, b_{(3)} \rangle \langle \varphi_{(3)}, \gamma^{-1}(b_{(1)}) \rangle \; (a \otimes 1^*)(b_{(2)} \otimes 1^*)(1 \otimes \varphi_{(2)})(1 \otimes \psi)$$

and by (i) $(a\otimes 1^*)(b_{(2)}\otimes 1^*) = ab_{(2)}\otimes 1^*$ and by (ii) $(1\otimes\varphi_{(2)})(1\otimes\psi) = 1\otimes\varphi_{(2)}\psi$ so again by (iii) the product above simplifies to

$$(a\otimes\varphi)(b\otimes\psi) = \sum_{(b),(\varphi)} \langle \varphi_{(1)}, b_{(3)}\rangle \langle \varphi_{(3)}, \gamma^{-1}(b_{(1)})\rangle \left(ab_{(2)}\otimes\varphi_{(2)}\psi\right).$$

Thus $\mu_{\mathcal{D}}$ is uniquely determined by the conditions. The unit in an algebra is uniquely determined, so $\eta_{\mathcal{D}}$ is unique, and it is easy to see that $1_{\mathcal{D}} = 1\otimes 1^* \in A\otimes B$ is the unit: for example for any $a\in A$, $\varphi\in B$ we have by the above formula

$$(a\otimes\varphi)(1\otimes 1^*) = \sum_{(\varphi)} \langle \varphi_{(1)}, 1\rangle \underbrace{\langle \varphi_{(3)}, \underbrace{\gamma^{-1}(1)}_{=1}\rangle}_{=\varepsilon_B(\varphi_{(3)})} \left(a1\otimes\varphi_{(2)}1^*\right)$$

$$\overset{\text{(H2') for } B}{=} \sum_{(\varphi)} \underbrace{\langle \varphi_{(1)}, 1\rangle}_{=\varepsilon(\varphi_{(1)})}\; a1\otimes\varphi_{(2)}1^*$$

$$\overset{\text{(H2') for } B}{=} a1\otimes\varphi 1^* \overset{\text{(H2) for } A \text{ and } B}{=} a\otimes\varphi.$$

The coproduct $\Delta_{\mathcal{D}}$ has to be a homomorphism of algebras, so using $a\otimes\varphi = (a\otimes 1^*)(1\otimes\varphi)$ and properties (i) and (ii), we get

$$\begin{aligned}\Delta_{\mathcal{D}}(a\otimes\varphi) &= \Delta_{\mathcal{D}}(a\otimes 1^*)\,\Delta_{\mathcal{D}}(1\otimes\varphi)\\ &= \Big(\sum_{(a)}(a_{(1)}\otimes 1^*)\otimes(a_{(2)}\otimes 1^*)\Big)\Big(\sum_{(\varphi)}(1\otimes\varphi_{(2)})\otimes(1\otimes\varphi_{(1)})\Big)\\ &= \sum_{(a),(\varphi)} (a_{(1)}\otimes\varphi_{(2)})\otimes(a_{(2)}\otimes\varphi_{(1)})\end{aligned}$$

so also $\Delta_{\mathcal{D}}$ is uniquely determined.
Similarly for $\varepsilon_{\mathcal{D}}$

$$\begin{aligned}\varepsilon_{\mathcal{D}}(a\otimes\varphi) &= \varepsilon_{\mathcal{D}}(a\otimes 1^*)\,\varepsilon_{\mathcal{D}}(1\otimes\varphi) = \varepsilon_A(a)\,\varepsilon_{B^{\mathrm{cop}}}(\varphi)\\ &= \varepsilon(a)\,\langle\varphi, 1\rangle.\end{aligned}$$

Finally for the antipode, recalling also that the antipode of the co-opposite Hopf algebra is the inverse of the original antipode, and that the antipode in the restricted dual is the transpose of the antipode, and that the antipode is an antihomomorphism of algebras:

$$\gamma_{\mathcal{D}}(a \otimes \varphi) = \gamma_{\mathcal{D}}(1 \otimes \varphi)\, \gamma_{\mathcal{D}}(a \otimes 1^*)$$

$$= \left(1 \otimes \gamma_{B^{cop}}(\varphi)\right)\left(\gamma(a) \otimes 1^*\right) = \left(1 \otimes \gamma_B^{-1}(\varphi)\right)\left(\gamma(a) \otimes 1^*\right)$$

[Here recall: $((\Delta \otimes id) \circ \Delta)(\gamma(a)) = \sum_{(a)} \gamma(a_{(3)}) \otimes \gamma(a_{(2)}) \otimes \gamma(a_{(1)})$

and $((\mu^* \otimes id) \circ \mu^*)(\gamma_B^{-1}(\varphi)) = \sum_{(\varphi)} \gamma_B^{-1}(\varphi_{(3)}) \otimes \gamma_B^{-1}(\varphi_{(2)}) \otimes \gamma_B^{-1}(\varphi_{(1)})$

$$= \sum_{(\varphi)} (\gamma^{-1})^*(\varphi_{(3)}) \otimes (\gamma^{-1})^*(\varphi_{(2)}) \otimes (\gamma^{-1})^*(\varphi_{(1)})$$]

$$\gamma_{\mathcal{D}}(a \otimes \varphi) = \sum_{(a),(\varphi)} \langle (\gamma^{-1})^*(\varphi_{(3)}), \gamma(a_{(1)}) \rangle \langle (\gamma^{-1})^*(\varphi_{(1)}), \gamma^{-1}(\gamma(a_{(3)})) \rangle \left(\gamma(a_{(2)}) \otimes (\gamma^{-1})^*(\varphi_{(2)})\right)$$

$$= \sum_{(a),(\varphi)} \langle \varphi_{(3)}, a_{(1)} \rangle \langle \varphi_{(1)}, \gamma^{-1}(a_{(3)}) \rangle \left(\gamma(a_{(2)}) \otimes (\gamma^{-1})^*(\varphi_{(2)})\right)$$

and thus indeed also $\gamma_{\mathcal{D}}$ is uniquely determined. □

We will later sketch the proof of the existence of a Hopf structure with the properties (i), (ii), (iii), (iv) on $\mathcal{D}(A,B) = A \otimes B$.

Before that we note that this construction gives braided Hopf algebras. In order for this to be literally the case, assume that A is a finite-dimensional Hopf algebra (with invertible antipode) and $B = A^*$. In this case denote briefly $\mathcal{D}(A, A^*) = \mathcal{D}(A)$.

Suppose that $(e_i)_{i=1}^d$ is a basis of A, and let $(\delta^i)_{i=1}^d$ denote the dual basis of A^* defined by
$$\langle \delta^i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}.$$
The evaluation map $ev_A : A^* \otimes A \to \mathbb{K}$ is this pairing $\varphi \otimes a \longmapsto \langle \varphi, a \rangle$ and the coevaluation map $coev_A : \mathbb{K} \longrightarrow A \otimes A^* \cong \mathrm{Hom}(A,A)$ is
$$coev_A(1) = \sum_{i=1}^d e_i \otimes \delta^i$$
which corresponds to $id_A \in \mathrm{Hom}(A,A)$, i.e.
$$\sum_{i=1}^d \langle \delta^i, b \rangle e_i = b \qquad \forall\, b \in A.$$

<u>Theorem</u> Let A be a finite-dimensional Hopf algebra with invertible antipode, and let $(e_i)_{i=1}^d$ and $(\delta^i)_{i=1}^d$ be dual bases of A and A^*.
Then the Drinfeld double $\mathcal{D}(A) = \mathcal{D}(A, A^*)$ is a braided Hopf algebra with a universal R-matrix
$$R = (\iota_A \otimes \iota_{A^*})(coev(1)) = \sum_{i=1}^d (e_i \otimes 1^*) \otimes (1 \otimes \delta^i).$$

<u>Proof</u>: First we check invertibility of $R \in \mathcal{D}(A) \otimes \mathcal{D}(A)$.
We claim that the inverse is (as it must be by general considerations – see exercises)
$$\bar{R} = \sum_{i=1}^d (\gamma(e_i) \otimes 1^*) \otimes (1 \otimes \delta^i).$$
Calculate for example
$$\bar{R}R = \Big(\sum_{j=1}^d (\gamma(e_j) \otimes 1^*) \otimes (1 \otimes \delta^j)\Big)\Big(\sum_{i=1}^d (e_i \otimes 1^*) \otimes (1 \otimes \delta^i)\Big)$$
$$= \sum_{i,j=1}^d (\gamma(e_j) e_i \otimes 1^*) \otimes (1 \otimes \delta^j \delta^i) \in \mathcal{D}(A) \otimes \mathcal{D}(A)$$
$$= A \otimes A^* \otimes A \otimes A^*.$$

Evaluate in the second and fourth components on $\alpha \in A$ and $b \in A$, respectively, to get

$$\bar{R}R \xmapsto[\text{on } \alpha \otimes b"]{\text{"evaluation}} \sum_{i,j=1}^{d} \langle 1^*, \alpha \rangle \langle \delta^j \delta^i, b \rangle \; \gamma(e_j) e_i \otimes 1$$

$$= \varepsilon(\alpha) \sum_{i,j=1}^{d} \sum_{(b)} \langle \delta^j, b_{(1)} \rangle \langle \delta^i, b_{(2)} \rangle \; \gamma(e_j)\, e_i \otimes 1$$

$$= \varepsilon(\alpha) \sum_{(b)} \gamma(b_{(1)})\, b_{(2)} \otimes 1 \;=\; \varepsilon(\alpha)\varepsilon(b)\; 1 \otimes 1 .$$

But this is clearly what the unit of $\mathcal{D}(A) \otimes \mathcal{D}(A)$

$$1_{\mathcal{D}} \otimes 1_{\mathcal{D}} = (1 \otimes 1^*) \otimes (1 \otimes 1^*)$$

would also evaluate to, so we see that

$$\bar{R}R = 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} .$$

Similarly one shows $R\bar{R} = 1_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.

To prove property (R1): $R \Delta_{\mathcal{D}}(x) R^{-1} = \Delta^{op}_{\mathcal{D}}(x)$ $\forall x \in \mathcal{D}(A)$, note that the set of elements

$$S = \{ x \in \mathcal{D}(A) \mid R \Delta_{\mathcal{D}}(x) = \Delta^{op}_{\mathcal{D}}(x) R \}$$

for which this holds is a subalgebra: if $x, y \in S$ then

$$R \Delta(xy) = R \Delta_{\mathcal{D}}(x) \Delta_{\mathcal{D}}(y) = \Delta^{op}_{\mathcal{D}}(x) R \Delta_{\mathcal{D}}(y)$$
$$= \Delta^{op}_{\mathcal{D}}(x) \Delta^{op}_{\mathcal{D}}(y) R = \Delta^{op}_{\mathcal{D}}(xy) R$$

and clearly $1_{\mathcal{D}} \in S$ since $\Delta_{\mathcal{D}}(1_{\mathcal{D}}) = 1_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.

It is therefore sufficient to check that $\alpha \otimes 1^* \in S$ $\forall \alpha \in A$ and $1 \otimes \varphi \in S$ $\forall \varphi \in A^*$, since elements of this form generate the entire Drinfeld double $\mathcal{D}(A)$ (as an algebra).

If $a \in A$ then

$$\Delta_{\mathcal{D}}^{op}(a \otimes 1^*)\, R = \sum_{(a)} \sum_{i=1}^{d} \Big((a_{(2)} \otimes 1^*) \otimes (a_{(1)} \otimes 1^*) \Big) \otimes \Big((e_i \otimes 1^*) \otimes (1 \otimes \delta^i) \Big)$$

$$\overset{(iii)}{=} \sum_{(a)} \sum_{i=1}^{d} (a_{(2)} e_i \otimes 1^*) \otimes (a_{(1)} \otimes \delta^i)$$

and

$$R\, \Delta_{\mathcal{D}}(a \otimes 1^*) = \sum_{(a)} \sum_{i=1}^{d} \Big((e_i \otimes 1^*) \otimes (1 \otimes \delta^i) \Big) \otimes \Big((a_{(1)} \otimes 1^*) \otimes (a_{(2)} \otimes 1^*) \Big)$$

$$\overset{(iv)}{=} \sum_{(a)} \sum_{i=1}^{d} \sum_{(\delta^i)} (e_i a_{(1)} \otimes 1^*) \otimes (a_{(3)} \otimes \delta^i_{(2)}) \langle \delta^i_{(1)}, a_{(4)} \rangle \langle \delta^i_{(3)}, \gamma^{-1}(a_{(2)}) \rangle .$$

These are elements in $\mathcal{D}(A) \otimes \mathcal{D}(A) = A \otimes A^* \otimes A \otimes A^*$ so to show their equality, we evaluate them in the second and fourth components on $b, c \in A$. The first evaluates to

$$\Delta_{\mathcal{D}}^{op}(a \otimes 1^*)\, R \longmapsto \sum_{(a)} \sum_{i=1}^{d} \langle 1^*, b \rangle \langle \delta^i, c \rangle \; a_{(2)} e_i \otimes a_{(1)}$$

$$= \varepsilon(b) \sum_{(a)} a_{(2)} c \otimes a_{(1)} .$$

and the second to

$$R\, \Delta_{\mathcal{D}}(a \otimes 1^*) \longmapsto \sum_{(a)} \sum_{i=1}^{d} \sum_{(\delta^i)} \langle \delta^i_{(1)}, a_{(4)} \rangle \langle \delta^i_{(3)}, \gamma^{-1}(a_{(2)}) \rangle \cdot \langle 1^*, b \rangle \langle \delta^i_{(2)}, c \rangle \; e_i a_{(1)} \otimes a_{(3)}$$

$$= \sum_{(a)} \sum_{i=1}^{d} \varepsilon(b) \langle \delta^i, a_{(4)} c\, \gamma^{-1}(a_{(2)}) \rangle \; e_i a_{(1)} \otimes a_{(3)}$$

$$= \varepsilon(b) \sum_{(a)} a_{(4)} c\, \gamma^{-1}(a_{(2)}) a_{(1)} \otimes a_{(3)}$$

$$\overset{(H3)\text{ for } A^{op}}{=} \varepsilon(b) \sum_{(a)} a_{(3)} c\, \varepsilon(a_{(1)}) 1 \otimes a_{(2)}$$

$$\overset{(H2')}{=} \varepsilon(b) \sum_{(a)} a_{(2)} c \otimes a_{(1)} .$$

These coincide, so $\Delta_{\mathcal{D}}^{op}(a \otimes 1^*) R = R\, \Delta_{\mathcal{D}}(a \otimes 1^*)$ $\forall a \in A$.
Similarly one shows $\Delta_{\mathcal{D}}^{op}(1 \otimes \varphi) R = R\, \Delta_{\mathcal{D}}(1 \otimes \varphi)$ $\forall \varphi \in A^*$.
Properties (R2) and (R3) are left as an exercise. $\square$

We now want to sketch the proof of the "existence" part of the theorem on Drinfeld doubles: that $\mathcal{D}(A,B)$ can be equipped with a Hopf algebra structure with the asserted properties (i), (ii), (iii), (iv). We have found formulas for the structural maps, so the goal is to show that the axioms (H1), (H1'), (H2), (H2') (H3), (H4), (H5), (H5'), (H6) hold for these maps.

The product in $\mathcal{D}(A,B)$ can by our formula be written as

$$\mu_{\mathcal{D}} : \underbrace{\mathcal{D}(A,B) \otimes \mathcal{D}(A,B)}_{A \otimes B \otimes A \otimes B} \longrightarrow \underbrace{\mathcal{D}(A,B)}_{A \otimes B}$$

$$\mu_{\mathcal{D}} = (\mu \otimes \Delta^*) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)$$

where $\tau : B \otimes A \longrightarrow A \otimes B$ is given by

$$\tau(\varphi \otimes b) = \sum_{(\varphi)} \sum_{(b)} \langle \varphi_{(1)}, b_{(3)} \rangle \langle \varphi_{(3)}, \gamma^{-1}(b_{(1)}) \rangle \; b_{(2)} \otimes \varphi_{(2)}$$

for $\varphi \in B$, $b \in A$.

The map $\tau : B \otimes A \longrightarrow A \otimes B$ satisfies the following.

Lemma We have the following equalities of linear maps:

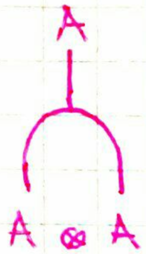
1°) $\tau \circ (\mathrm{id}_B \otimes \mu) = (\mu \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau) \circ (\tau \otimes \mathrm{id}_A) : B \otimes A \otimes A \to A \otimes B$

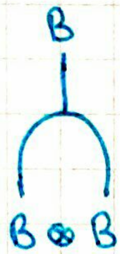
2°) $\tau \circ (\Delta^* \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes \Delta^*) \circ (\tau \otimes \mathrm{id}_B) \circ (\mathrm{id}_B \otimes \tau) : B \otimes B \otimes A \to A \otimes B$.

Proof: See typeset lecture notes, Lemma 4.19. $\square$

This lemma helps verify associativity of $\mu_{\mathcal{D}}$. The argument becomes clearest in graphical notation — for a formal version, see typeset notes.

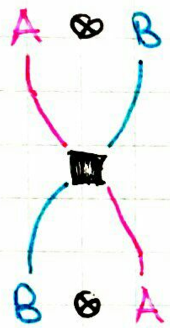
Denote A by a pink vertical line and B by a blue vertical line, and the products $\mu : A \otimes A \longrightarrow A$ and $\Delta^* : B \otimes B \rightarrow B$ by





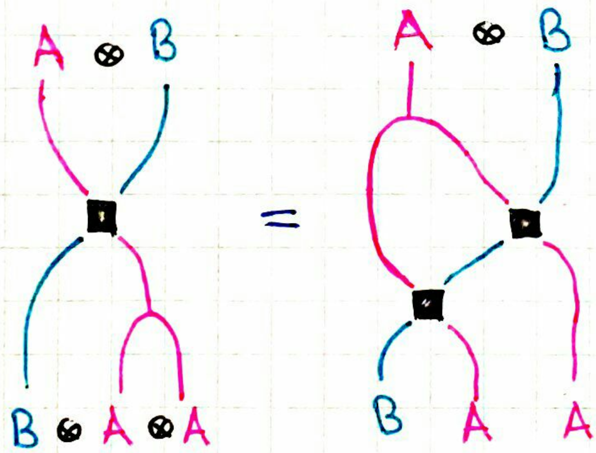
(interpret horizontally adjacent vertical lines as a tensor product of the corresponding vector spaces, and read the diagrams as linear maps from the bottom to the top.

Denote moreover the map $\tau : B \otimes A \longrightarrow A \otimes B$ by

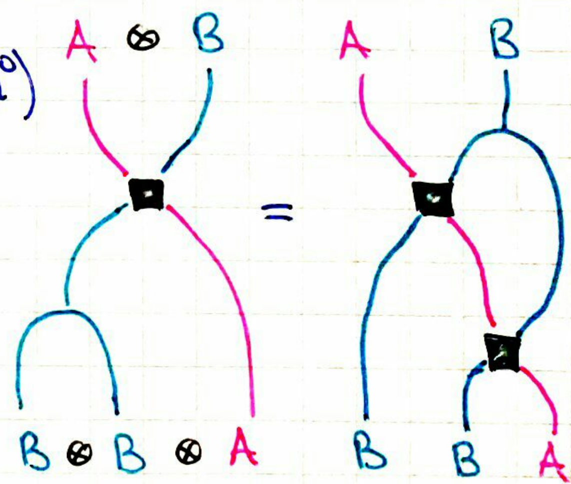


The equalities of linear maps in the lemma above can then be graphically represented as

1°)

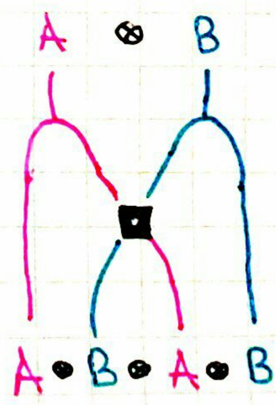


2°)

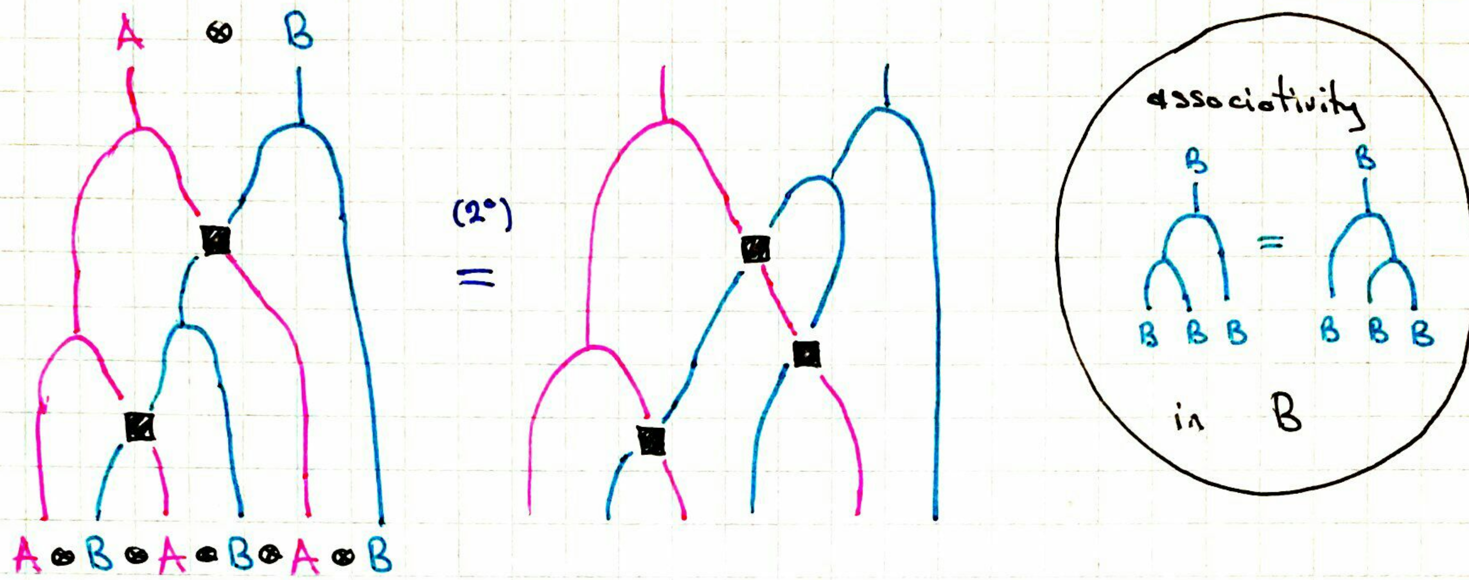


The product $\mu_{\mathcal{D}}$ in the Drinfeld double $\mathcal{D}(A,B) = A \otimes B$ is $\mu_{\mathcal{D}} = (\mu \otimes \Delta^*) \circ (id_A \otimes \tau \otimes id_B)$, graphically represented as:

$\mu_{\mathcal{D}} =$

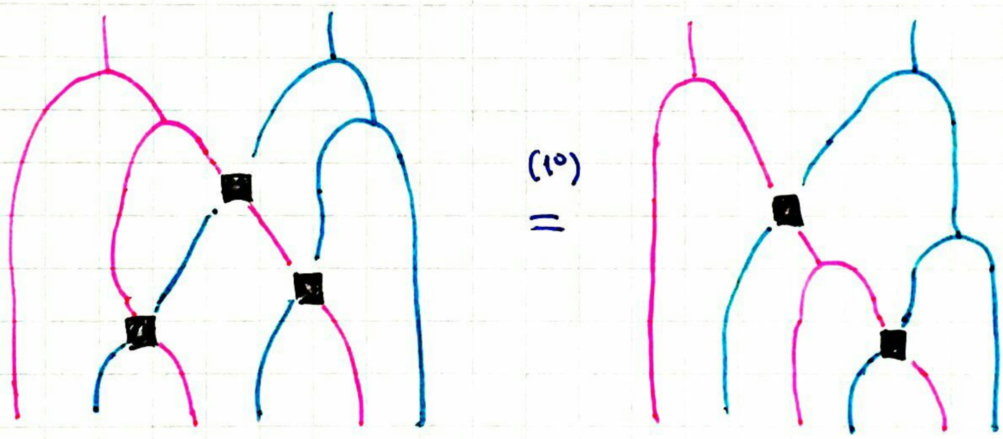


The associativity of $\mu_{\mathcal{D}}$ is checked graphically as follows: start from the diagram for the composition $\mu_{\mathcal{D}} \circ (\mu_{\mathcal{D}} \otimes id_{\mathcal{D}}) : A \otimes B \otimes A \otimes B \otimes A \otimes B \longrightarrow A \otimes B$



associativity in A and B

=



and notice that after using (2°), associativity for μ and Δ^* and (1°), one obtains the diagram for the composition $\mu_{\mathcal{D}} \circ (id_{\mathcal{D}} \otimes \mu_{\mathcal{D}})$.

This proves (H1) in $\mathcal{D}(A,B)$. We leave the details of the other axioms to the reader.

THE QUANTUM GROUP $\mathcal{U}_q(\mathfrak{sl}_2)$

There is no general definition of what a "quantum group" is, but the term usually refers to a Hopf algebra with some (or most) of the following features:

- non-commutativity and non-cocommutativity
- braidedness (perhaps in a slightly relaxed sense: there may not be a universal R-matrix, but the category of representations nevertheless has braiding)
- dependence on a "deformation parameter", usually denoted by q.

The prototypical examples are obtained as "q-deformations" of (universal enveloping algebras of) simple Lie algebras. When such deformations were first discovered by Drinfeld and Jimbo, they were considered surprising, since simple Lie algebras had seemed very rigid (they are in particular classified in 4 infinite series and 5 exceptional cases by purely combinatorial data encoded e.g. in the Cartan matrix or Dynkin diagram).

In this lecture we discuss the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, which is a "$q$-deformation" of the simplest simple Lie algebra: $\mathfrak{sl}_2$. In the literature, the deformation parameter is interpreted in various ways in different contexts, for example

- $q \in \mathbb{C} \setminus \{+1, 0, -1\}$ a complex number (cases with q a root of unity behave very differently from the generic case)
- q an indeterminate: one works over the ground field $\mathbb{K} = \mathbb{C}(q)$ of rational functions of q
- $q = \exp(\hbar)$ with $\hbar$ a formal parameter: one works over the ring $\mathbb{C}[[\hbar]]$ of power series in $\hbar$.

In all interpretations, one should recover the undeformed case as $q \longrightarrow 1$ (or $\hbar \rightarrow 0$).

We focus on the first interpretation: q will be a fixed complex number. This will display the special role of roots of unity in the most clear manner possible.

For any other simple Lie algebra $\mathfrak{g}$ it is possible to form the associated quantum group $\mathcal{U}_q(\mathfrak{g})$ in a similar manner — although the details get much more complicated.

The Lie algebra: $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ is the 3-dimensional (complex) simple Lie algebra with basis $E, H, F \in \mathfrak{sl}_2$ and Lie bracket $[\cdot,\cdot] : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2$ (bilinear, antisymmetric, satisfying Jacobi identity) determined by

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H.$$

Its standard representation (two-dimensional) $\rho_{std} : \mathfrak{sl}_2 \longrightarrow \mathrm{End}(\mathbb{C}^2)$ is given by

$$\rho_{std}(H) = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_{std}(E) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \rho_{std}(F) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ is an (associative) algebra such that representations of the Lie algebra $\mathfrak{sl}_2$ correspond exactly to representations of the (associative) algebra $\mathcal{U}(\mathfrak{sl}_2)$: it has generators E, H, F subject to relations

$$HE - EH = 2E, \qquad HF - FH = -2F, \qquad EF - FE = H.$$

As usually with universal enveloping algebras of Lie algebras, $\mathcal{U}(\mathfrak{sl}_2)$ has a natural Hopf algebra structure: it is determined by declaring that the generators E, H, F are primitive elements:

$$\Delta(E) = E \otimes 1 + 1 \otimes E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

and it follows that $\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$ and

$$\gamma(E) = -E, \qquad \gamma(H) = -H, \qquad \gamma(F) = -F.$$

The Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$

The "q-deformation" of $\mathcal{U}(\mathfrak{sl}_2)$ is in some vague sense obtained by using a generator "$K = q^H$" instead of the Cartan element $H \in \mathfrak{sl}_2$.

Fix a parameter $q \in \mathbb{C} \setminus \{+1, 0, -1\}$.
The algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by E, F, K, K^{-1} subject to relations

$$KK^{-1} = 1 = K^{-1}K, \qquad KE = q^2 EK, \quad KF = q^{-2} FK$$
$$EF - FE = \frac{1}{q - q^{-1}} (K - K^{-1}).$$

We make $\mathcal{U}_q(\mathfrak{sl}_2)$ into a Hopf algebra: the coproducts of the generators are defined as

$$\Delta(K) = K \otimes K, \qquad (K \text{ is grouplike})$$
$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

It is straightforward to verify that

$$\Delta(K)\Delta(E) = q^2 \cdot \Delta(E)\Delta(K), \qquad \Delta(K)\Delta(F) = q^{-2} \cdot \Delta(F)\Delta(K)$$
$$\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = \frac{1}{q - q^{-1}} (\Delta(K) - \Delta(K^{-1}))$$

so indeed there exists a (unique, of course) homomorphism of algebras $\Delta : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$ with these values on generators.
If a counit satisfying (H2), $\varepsilon : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathbb{C}$, were to exist, it is easy to see that

$$\varepsilon(K) = 1 \qquad \text{(as always for grouplike elements)}$$
$$\varepsilon(E) = 0, \qquad \varepsilon(F) = 0.$$

Again $\varepsilon(K)\varepsilon(E) = q^2 \cdot \varepsilon(E)\varepsilon(K)$ etc. so an algebra homomorphism $\varepsilon : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathbb{C}$ with these values on generators exists.

Similarly, if an antipode satisfying (H3) exists, then from the coproducts and counits above one easily gets

$$\gamma(K) = K^{-1} , \quad \gamma(E) = -EK^{-1} , \quad \gamma(F) = -KF .$$

Now $\gamma(E)\gamma(K) = q^2 \gamma(K)\gamma(E)$ etc. so there exists a homomorphism of algebras $\gamma : \mathcal{U}_q(\underline{sl}_2) \longrightarrow \mathcal{U}_q(\underline{sl}_2)^{op}$ determined by these.

It is enough to check the axioms on the generators (the sets of elements on which each axiom holds is a subalgebra), which is routine. So indeed there exists a Hopf algebra structure as above on $\mathcal{U}_q(\underline{sl}_2)$ and it is uniquely determined by the formulas for $\Delta(K), \Delta(E), \Delta(F)$.

<u>Realization of $\mathcal{U}_q(\underline{sl}_2)$ via a Drinfeld double</u>

$\mathcal{U}_q(\underline{sl}_2)$ is essentially a Drinfeld double...

More precisely, assume that $q \in \mathbb{C} \setminus \{+1, 0, -1\}$ is not a root of unity: $q^n \neq 1 \quad \forall n \in \mathbb{Z} \setminus \{0\}$. Let $Q = q^2$ and consider the Hopf algebra H_Q generated (as an algebra) by elements a, a^{-1}, b subject to relations

$$a a^{-1} = 1 = a^{-1} a , \qquad ab = Q \cdot ba$$

with the Hopf algebra structure determined by the coproducts of generators

$$\Delta(a) = a \otimes a , \qquad \Delta(b) = a \otimes b + b \otimes 1 .$$

H can be shown (see typeset notes for details, here the assumption Q not a root of unity is relevant) that the restricted dual H_Q° contains a Hopf subalgebra isomorphic to H_Q itself, with generators $\bar{a}, \bar{b} \in H_Q^\circ$. Let $\mathcal{D}(H_Q, H_Q)$ denote the associated Drinfeld double. Finally, in $\mathcal{D}(H_Q, H_Q)$ the ideal J generated by the element $(a \otimes 1^*)(1 \otimes \bar{a}) - 1_{\mathcal{D}}$ is a Hopf ideal, and the quotient Hopf algebra is isomorphic to $\mathcal{U}_q(\underline{sl}_2)$:

$$\mathcal{D}(H_{q^2}, H_{q^2}) / J \cong \mathcal{U}_q(\underline{sl}_2).$$

Because of this relationship with a Drinfeld double, one should expect $\mathcal{U}_q(\underline{sl}_2)$ to be braided (or at least something close to braided). Indeed, although there is no universal R-matrix in $\mathcal{U}_q(\underline{sl}_2) \otimes \mathcal{U}_q(\underline{sl}_2)$ in a literal sense, the category of finite dimensional representations of $\mathcal{U}_q(\underline{sl}_2)$ is braided, i.e. we have intertwining maps (for any finite-dim. reps V, W)

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V$$

such that

$$c_{\mathbb{C} \otimes V} = id_V : V \to V, \qquad c_{V \otimes \mathbb{C}} = id_V : V \to V$$

$$c_{U \otimes V, W} = (c_{U,W} \otimes id_V) \circ (id_U \otimes c_{V,W}) : U \otimes V \otimes W \to W \otimes U \otimes V$$

$$c_{U, V \otimes W} = (id_V \otimes c_{U,W}) \circ (c_{U,V} \otimes id_W) : U \otimes V \otimes W \to V \otimes W \otimes U.$$

The construction of these $c_{V,W}$ works just as if $\mathcal{U}_q(\underline{sl}_2)$ had a universal R-matrix (see typeset notes for details).

Representations of $\mathcal{U}_q(\underline{\mathfrak{sl}}_2)$ in the generic case

Our next task is to classify concretely the finite-dimensional representations of $\mathcal{U}_q(\underline{\mathfrak{sl}}_2)$. There is a big difference depending on whether q is a root of unity or not. As a vague summary:

$q \notin e^{i\pi\mathbb{Q}}$: The representation theory of $\mathcal{U}_q(\underline{\mathfrak{sl}}_2)$ is almost the same as that of the Lie algebra $\underline{\mathfrak{sl}}_2$. In particular we have complete reducibility: all finite-dim. representations are direct sums of irreducible representations, which one can list very concretely.

"generic case"

$q \in e^{i\pi\mathbb{Q}}$: At roots of unity, semisimplicity breaks down: there are finite-dimensional representations which are not irreducible while still being indecomposable in the sense that they can not be written as direct sums of nontrivial subrepresentations.

"root of unity case"

We will only consider the generic case, so from here on assume that

$$q \notin e^{i\pi\mathbb{Q}} \qquad \text{i.e.} \qquad q^n \neq 1 \quad \forall n \in \mathbb{Z}\setminus\{0\}.$$

We start by analyzing a finite-dimensional representation V, and from the analysis derive first the classification of irreducible representations. The proof of complete reducibility will then be the next topic.

Suppose V is a finite-dimensional representation of $\mathcal{U}_q(\underline{\mathfrak{sl}}_2)$ (with $q \notin e^{i\pi\mathbb{Q}}$). If $V \neq \{0\}$ then there exists an eigenvector $v' \in V \setminus \{0\}$ of K,

$$K.v' = \lambda' \cdot v' \qquad (\lambda' \in \mathbb{C} \setminus \{0\} \text{ since } K \text{ is invertible}).$$

Then note that the vector $E.v'$ satisfies

$$K.(E.v') = KE.v' = q^2 \, EK.v' = \lambda' \cdot q^2 (E.v)$$

so that either $E.v$ is an eigenvector of K with eigenvalue $\lambda' \cdot q^2$ or $E.v' = 0$. Continuing similarly, for any $m \in \mathbb{N}$ the vector $E^m.v'$ satisfies

$$K.(E^m.v') = \lambda' \cdot q^{2m} (E^m.v').$$

The numbers $\lambda', \lambda' \cdot q^2, \lambda' \cdot q^4, \lambda' \cdot q^6, \ldots$ are all distinct since $\lambda' \neq 0$ and $q^n \neq 1$ for $n \in \mathbb{Z} \setminus \{0\}$. Corresponding eigenvectors would be linearly independent, so because $\dim(V) < \infty$, necessarily for some m we have $E^m.v' = 0$. If m is the smallest such positive integer, then $w_0 = E^{m-1}.v'$ is a non-zero vector satisfying

$$E.w_0 = 0 \qquad \text{and} \qquad K.w_0 = \lambda \cdot w_0$$

$$(\lambda = \lambda' \cdot q^{2m-2}).$$

Such a vector is called a highest weight vector.

Now define $w_j := F^j.w_0$ for $j = 0, 1, 2, \ldots$ and observe that these vectors again satisfy

$$K.w_j = KF^j.w_0 = q^{-2j} F^j K.w_0 = q^{-2j} \cdot \lambda \cdot F^j.w_0$$
$$= \lambda q^{-2j} \cdot w_j,$$

so that they are either eigenvectors of K or zero. As above, $\lambda \neq 0$, $q \notin e^{i\pi\mathbb{Q}}$, and $\dim(V) < \infty$ imply that for some $d \in \mathbb{N}^*$, $w_d = 0$ but $w_{d-1} \neq 0$.

We now claim that $W = \mathrm{span}\{w_0, w_1, \ldots, w_{d-1}\}$ is a d-dimensional subrepresentation of V. Dimensionality $\dim(W) = d$ is clear, since $w_0, w_1, \ldots, w_{d-1}$ are eigenvectors with different eigenvalues. By construction also

$$F.W \subset W \quad \text{and} \quad K.W \subset W$$

so in order to prove that $W \subset V$ is an invariant subspace and therefore a subrepresentation, we must show that $E.W \subset W$. For that, calculate

$$E.w_j = EF.w_{j-1} = \left(FE + \frac{1}{q-q^{-1}}(K-K^{-1})\right).w_{j-1}$$

$$= \frac{1}{q-q^{-1}}\left(\lambda q^{2-2j} - \lambda^{-1} q^{2j-2}\right) w_{j-1} + FE.w_{j-1}$$

and continue similarly with the last term

$$FE.w_{j-1} = FEF.w_{j-2} = F.\left(FE + \frac{1}{q-q^{-1}}(K-K^{-1})\right).w_{j-2}$$

$$= \frac{1}{q-q^{-1}}\left(\lambda q^{4-2j} - \lambda^{-1} q^{2j-4}\right) w_{j-1} + F^2E.w_{j-2}$$

and so on. The result after j iterations is

$$E.w_j = \frac{1}{q-q^{-1}}\left(\lambda\cdot(q^{2-2j} + \cdots + q^{-2} + 1) + \lambda^{-1}\cdot(q^{2j-2} + \cdots + q^2 + 1)\right) w_{j-1} + \underbrace{F^j E w_0}_{=0}$$

$$= \frac{1}{q-q^{-1}}\left(\lambda \cdot \frac{1-q^{-2j}}{1-q^{-2}} + \lambda^{-1} \cdot \frac{1-q^{2j}}{1-q^2}\right) w_{j-1}.$$

In particular we see that $E.W \subset W$, so indeed W is a subrepresentation.

Denote furthermore

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} \cdot \frac{1-q^{2n}}{1-q^2} = q^{1-n} [\![n]\!]_{q^2}.$$

Then the above calculation can be simplified to

$$E.w_j = [j]_q \cdot \left(\frac{\lambda q^{1-j} + \lambda^{-1} \cdot q^{j-1}}{q - q^{-1}}\right) \cdot w_{j-1}.$$

Substituting $j=d$ we get

$$0 = E.0 = E.w_d = \underbrace{[d]_q}_{\neq 0} \cdot \left(\frac{\lambda \cdot q^{1-d} + \lambda^{-1} q^{d-1}}{q-q^{-1}} \right) \cdot \underbrace{w_{d-1}}_{\neq 0}$$

which implies $\lambda \cdot q^{1-d} + \lambda^{-1} \cdot q^{d-1} = 0$. The two solutions to $\alpha + \alpha^{-1} = 0$ are $\alpha = \pm 1$, so we get $\lambda \cdot q^{1-d} = \pm 1$, i.e., $\lambda = \pm q^{d-1}$. Therefore we can simplify further

$$K.w_j = \lambda \cdot q^{-2j} \cdot w_j = \pm q^{d-1-2j} \, w_j$$

$$E.w_j = [j]_q \cdot \frac{\pm q^{d-j} \mp q^{j-d}}{q-q^{-1}} \, w_{j-1} = \pm [j]_q \cdot [d-j]_q \cdot w_{j-1}.$$

Note that if $V \neq \{0\}$ is an irreducible finite-dimensional subrepresentation, then the non-zero subrepresentation $W \subset V$ must be the entire representation, $V = W$. Therefore we have the following classification of all irreducible representations:

Theorem For $q \notin e^{i\pi\mathbb{Q}}$, any finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ is isomorphic to some W_d^σ, $\sigma \in \{+1,-1\}$, $d \in \mathbb{N}^*$, where W_d^σ is the representation with basis $w_0, w_1, \dots, w_{d-1}$ and action of generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ given by

$$K.w_j = \sigma \cdot q^{d-1-2j} \, w_j$$

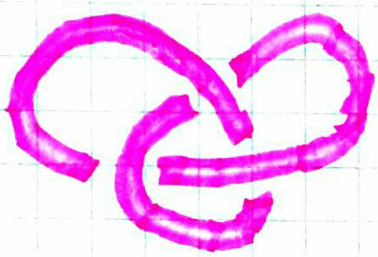
$$F.w_j = \begin{cases} w_{j+1} & \text{if } j < d-1 \\ 0 & \text{if } j = d-1 \end{cases}$$

$$E.w_j = \begin{cases} 0 & \text{if } j = 0 \\ \sigma \cdot [j]_q [d-j]_q \, w_{j-1} & \text{if } j > 0. \end{cases}$$

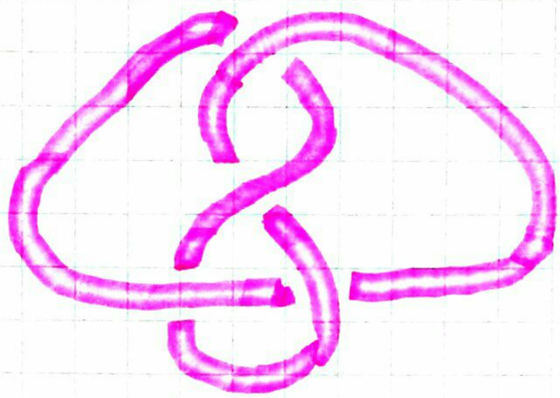
HOW TO CONSTRUCT KNOT INVARIANTS USING HOPF ALGEBRAS ?

KNOTS (AND LINKS AND TANGLES)

Two examples of what knots may look like are:



"trefoil knot"

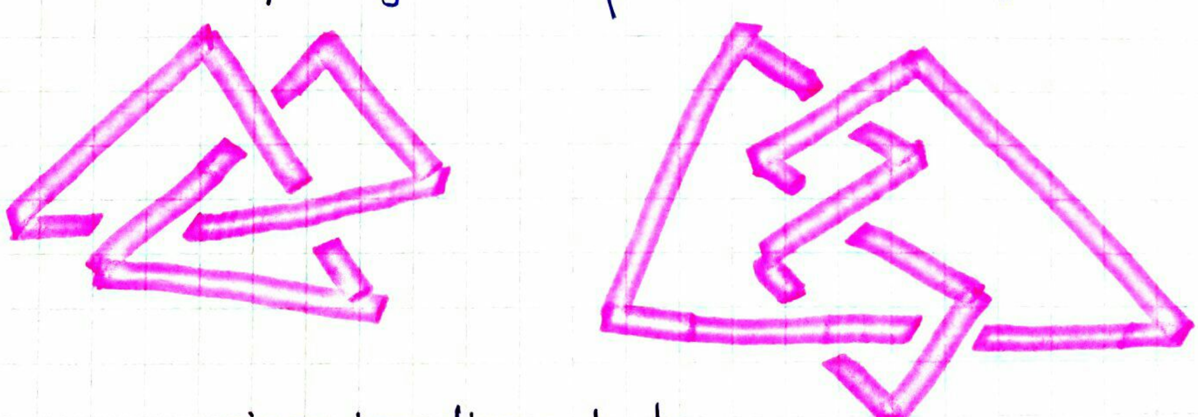


"figure eight knot"

Formally, a knot is the embedded image of the circle S^1 in space $\mathbb{R}^3$, i.e. the image

$$K = f(S^1) \subset \mathbb{R}^3$$

$f: S^1 \to \mathbb{R}^3$ is continuous and injective. In order to avoid pathological knots, one may require that f is smooth ($f \in C^\infty(S^1, \mathbb{R}^3)$), or as is common, f is piecewise linear.



piecewise linear knots

We think of knots made of string, and want to allow various ways on untangling (or retangling) the knot. This is captured by the notion of isotopy.

<u>Def</u>: Two (piecewise linear / smooth) knots K and $\tilde{K}$ are isotopic if there exists an isotopy of $\mathbb{R}^3$, i.e. a continuous (piecewise linear / smooth) function $h:[0,1]\times\mathbb{R}^3 \to \mathbb{R}^3$ such that $h(0,x)=x$ $\forall x\in\mathbb{R}^3$ and for all $t\in[0,1]$ the map $x\mapsto h(t,x)$ is a homeomorphism $\mathbb{R}^3\to\mathbb{R}^3$, and $h(1,K)=\tilde{K}$.

Isotopy is an equivalence relation, and knot theory aims at classifying knots up to isotopy, i.e., describing the equivalence classes $[K]$ of knots under this equivalence relation. This corresponds to the idea of untangling string.

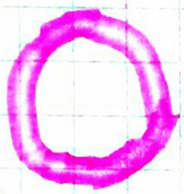
<u>Isotopy invariants</u>

If two knots K and $\tilde{K}$ are isotopic, this can be proven by concretely exhibiting an isotopy h between them. The converse is less obvious: how does one prove that two knots are not isotopic?

One practical way is to assign to any knot K some quantity I_K such that whenever K_1 and K_2 are isotopic, one has $I_{K_1}=I_{K_2}$. Then at least if $I_K\neq I_{\tilde{K}}$ we can conclude that K and $\tilde{K}$ are not isotopic. Such quantities are called isotopy invariants. The goal is to choose isotopy invariants that are as easy as possible to compute, and which distinguish as many knots from each other as possible.

Example Setting $I_K = \text{const.}$ (the same for all knots K) one gets an isotopy invariant, but this is useless because we never have $I_K \neq I_{\tilde{K}}$ even for non-isotopic knots.

Example Setting $I_K = \pi_1(\mathbb{R}^3 \setminus K)$, the fundamental group of the complement of the knot K, one gets an isotopy invariant (the isomorphism type of the fundamental group of the knot complement does not change under isotopy). For example, for



"unknot"

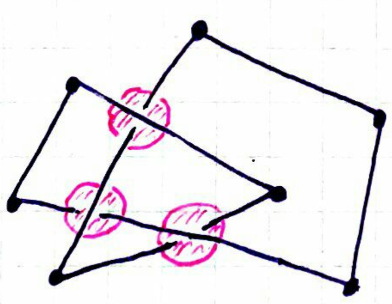
one has $I_O = \pi_1(\mathbb{R}^3 \setminus O) = \mathbb{Z}$. Now whenever $I_K = \pi_1(\mathbb{R}^3 \setminus K) \not\cong \mathbb{Z}$, one can conclude that K is not isotopic to the unknot.

The problem is that even if one can relatively easily find some generators and relations description of $\pi_1(\mathbb{R}^3 \setminus K)$, deciding the isomorphism of two different such descriptions is again a hard problem. Thus one would like to find isotopy invariants that are simpler objects to work with.

One extremely useful first idea is to represent knots by diagrams in the plane: this makes the classification of knots up to isotopy a more combinatorial task.

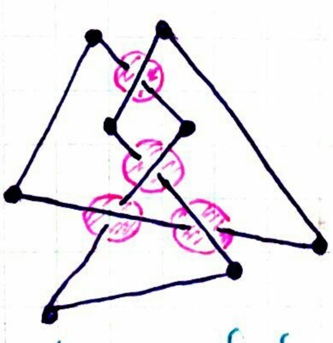
Knot diagrams

Choose a plane P in $\mathbb{R}^3$, and a projection $p : \mathbb{R}^3 \to P \cong \mathbb{R}^2$. Consider the projected image $p(K) \subset P \cong \mathbb{R}^2$ of the knot K. Generically, a point $z \in P$ is not in the image, $p^{-1}(z) \cap K = \phi$. Generically a point $z \in p(K)$ in the image has exactly one corresponding point on the knot, i.e. $\#(p^{-1}(z) \cap K) = 1$. Points z such that $\#(p^{-1}(z) \cap K) \geq 2$ are called crossings, and in order to keep all relevant information about the (isotopy class) of the knot K, we should supplement each crossing with the information about the ordering of the points $p^{-1}(z)$ in the direction that is not seen by the projection. Generically for crossings we have $\#(p^{-1}(z) \cap K) = 2$, and then there are two possible orderings. A knot diagram for K is a projection $p : \mathbb{R}^3 \to P \cong \mathbb{R}^2$ with finitely many crossings $(\#\{z \in P \mid \#p^{-1}(z) > 1\}$ finite$)$ and $\#(p^{-1}(z) \cap K) \leq 2$ $\forall z \in P$, together with a labeling of the crossings by the two possible orderings. We do not do this more formally — the following examples should suffice:

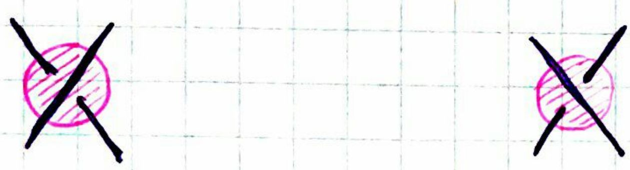


a diagram of trefoil knot

: crossing



a diagram of figure-8 knot



the two possible orderings for a crossing

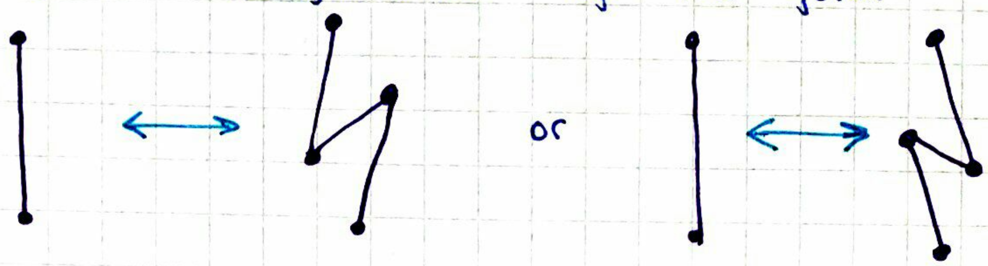
We say that a knot diagram (in $\mathbb{R}^2$) of a piecewise linear knot is generic if moreover all vertices (endpoints of the linear pieces) and all crossings have different heights (y-components in $\mathbb{R}^2$).

It is not hard to show that any piecewise linear knot has generic knot diagrams and that a generic knot diagram determines the isotopy class of the knot it represents. Moreover, the isotopy of knots can be determined as follows (in the combinatorial spirit of Reidemeister). Two generic knot diagrams $D, \tilde{D}$ are isotopic in $\mathbb{R}^2$ if and only if they can be obtained from one another by a finite sequence of transformations of the following types:

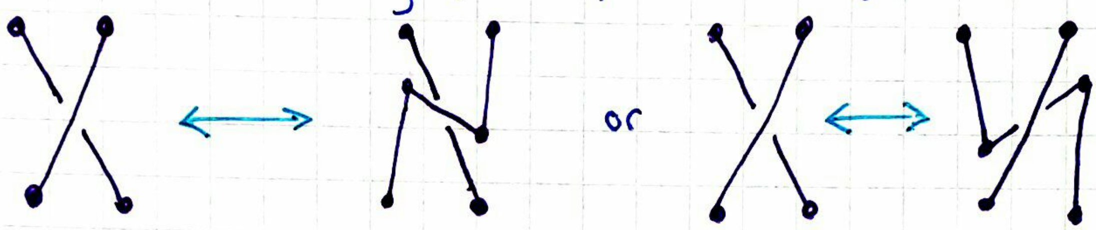
(A): a generic isotopy of the plane ($h : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(0,z) = z \ \ \forall z \in \mathbb{R}^2$, $h(1, D) = \tilde{D}$, and $h(t, D)$ is a generic knot diagram for all $t \in [0,1]$)

(B): an isotopy of the plane interchanging the order of two vertices, two crossings, or a vertex and a crossing, with respect to their heights (y-coordinates)

(C): a local transformation of the form



(D): a local transformation in a neighborhood of a crossing, of the form

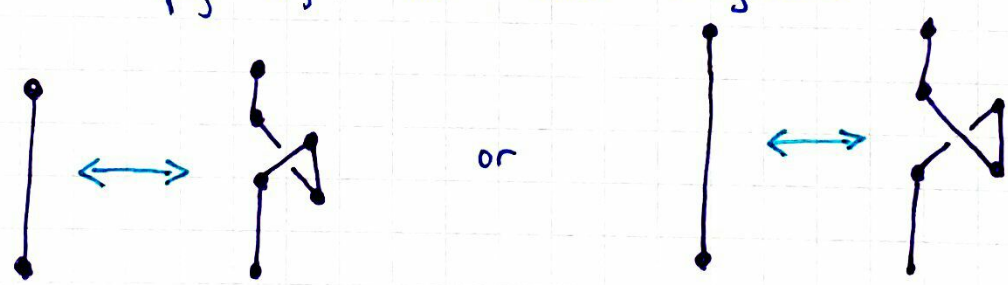


or a similar one for the other crossing type.

Finally, two generic knot diagrams $D, \tilde{D}$ represent isotopic knots if and only if they can be obtained from one another by a finite sequence of the following "Reidemeister transformations":

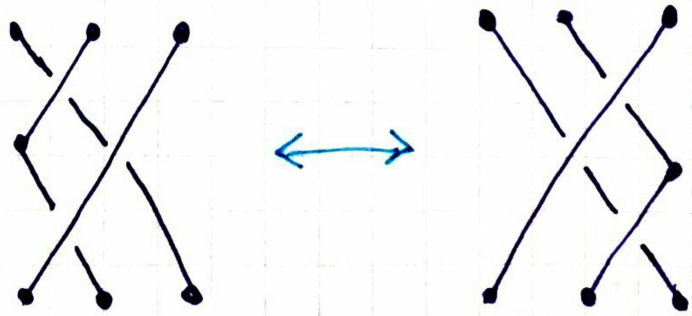
- an isotopy of the knot diagrams

(I):



(II):

(III):



What is usually meant by a "knot invariant", then, is a quantity I_D assigned to each generic knot diagram D so that $I_{D_1} = I_{D_2}$ whenever D_1 and D_2 are related by the above transformations.

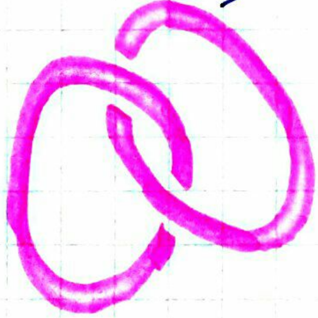
Remarks 1°) It is sufficient to check that the quantity remains unchanged under each type of tranformation: (A), (B), (C), (D), (I), (II), (III), thus reducing the invariance to an essentially combinatorial verification.

2°) From a "knot invariant" $D \mapsto I_D$ of this type we then get an actual isotopy invariant of knots by choosing any generic diagram D for a knot K.

Before we enter the construction of knot invariants, we briefly discuss two harmless generalizations and three useful refinements.

Links and tangles

Instead of the embedded image of just one circle $f: S^1 \to \mathbb{R}^3$ one may consider links which are embedded images of disjoint unions of finitely many circles, e.g.

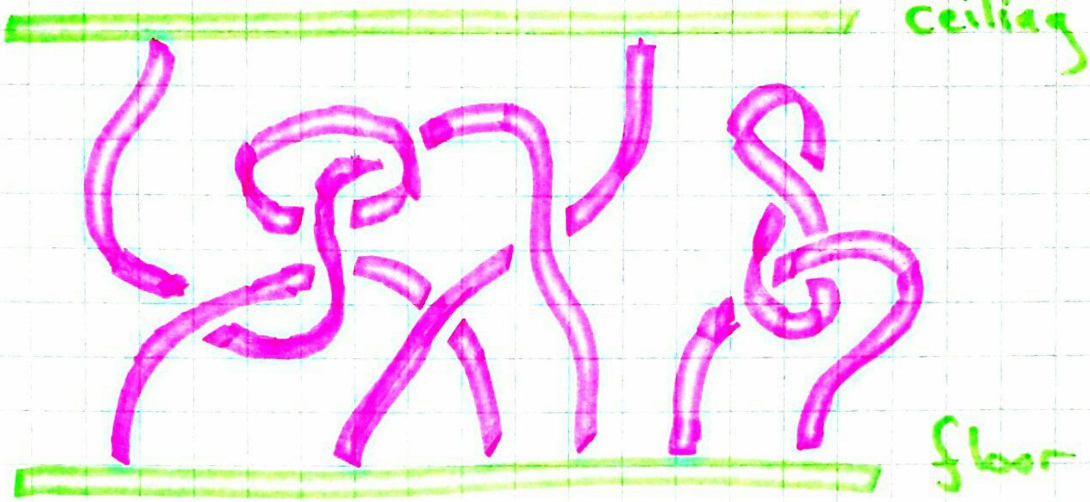


"the Hopf link"

surprise: Hopf was in fact working in geometry and topology!

[Heinz Hopf 1894-1971]

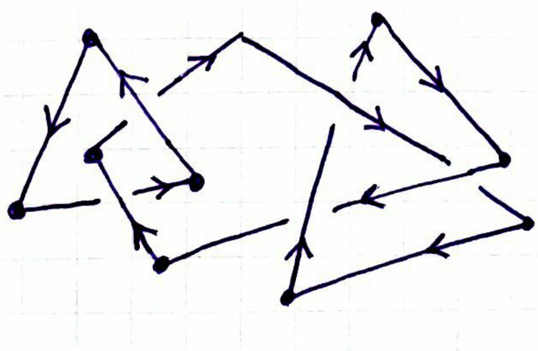
We could also consider even more general *tangles*, where also embedded images of closed intervals with endpoints in a "floor" or a "ceiling" are allowed, such as



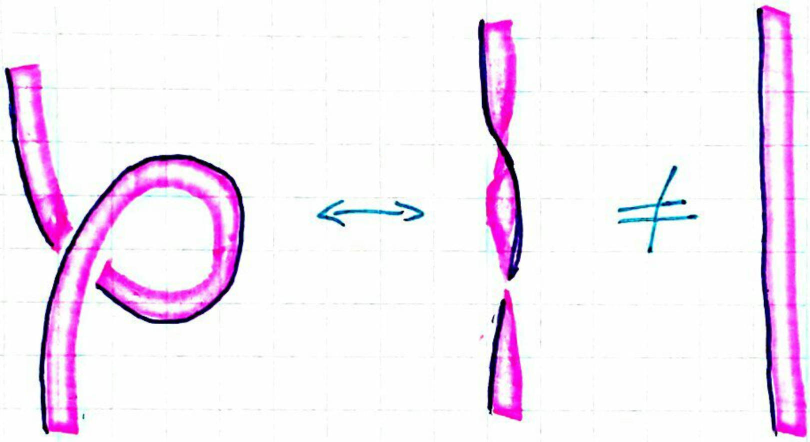
(a tangle)

Oriented, framed, and colored knots (or links or tangles)

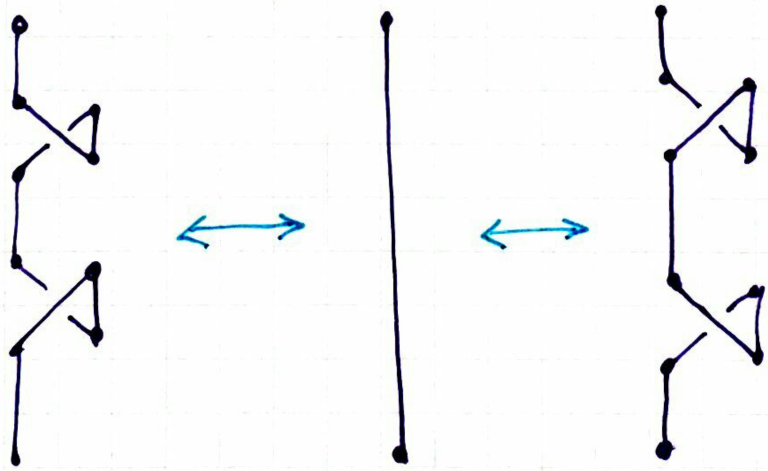
When knots are *oriented*, we keep track of the orientation of the image of S^1, and allow only isotopies that respect orientation. For links and tangles, each component is assigned its own orientation. The orientation is denoted by arrows in the diagram, e.g.



A framed knot is a knot together with a vector field on the knot that is nowhere tangent to the knot. It can be thought of as making the string into a ribbon, which has a small nonzero "width" in the direction of the vector field. The main difference in the theory is that Reidemeister move (I) is no longer valid, since



We may find representative diagrams for framed knots in which the vector field always points out from the plane of projection, and Reidemeister move (I) then has to be replaced by:

(I'): 

Finally, a knot, link, or a tangle is colored, if its each component has an assigned "color" from some set of allowed colors. We use different representations of a given Hopf algebra as the colors.

CONSTRUCTION OF KNOT INVARIANTS

Assume that H is a Hopf algebra, $(V_\alpha)_{\alpha \in \mathcal{A}}$ is some set of its finite-dim. representations which

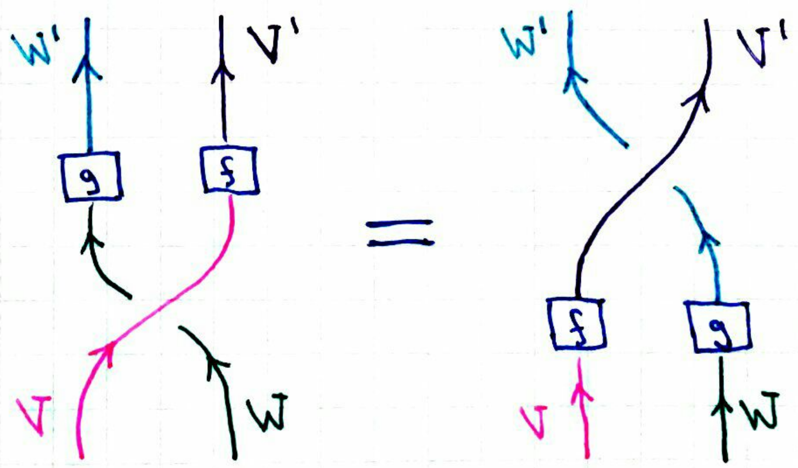
- contains the trivial representation: $V_{\alpha_0} \cong \mathbb{C}$ for some $\alpha_0 \in \mathcal{A}$.
- is stable under tensor products: $\forall \alpha, \beta \ \exists \gamma$
$$V_\alpha \otimes V_\beta \cong V_\gamma$$
- has braidings
$$c_{V_\alpha, V_\beta} : V_\alpha \otimes V_\beta \longrightarrow V_\beta \otimes V_\alpha$$
that satisfy:
  - c_{V_α, V_β} is H-intertwining isomorphism

(i): $c_{\mathbb{C}, V_\alpha} = \mathrm{id}_{V_\alpha} = c_{V_\alpha, \mathbb{C}}$

(ii): $c_{V_\alpha \otimes V_\beta, V_\gamma} = (c_{V_\alpha, V_\gamma} \otimes \mathrm{id}_{V_\beta}) \circ (\mathrm{id}_{V_\alpha} \otimes c_{V_\beta, V_\gamma})$

(iii): $c_{V_\alpha, V_\beta \otimes V_\gamma} = (\mathrm{id}_{V_\beta} \otimes c_{V_\alpha, V_\gamma}) \circ (c_{V_\alpha, V_\beta} \otimes \mathrm{id}_{V_\gamma})$

  - naturality: whenever $f : V \to V'$, $g : W \to W'$ are H-intertwining, we have
$$(g \otimes f) \circ c_{V,W} = c_{V', W'} \circ (f \otimes g).$$



▸ is stable under taking duals: $\forall \alpha \; \exists \alpha'$ s.t.

$$V_\alpha^* \cong V_{\alpha'}$$

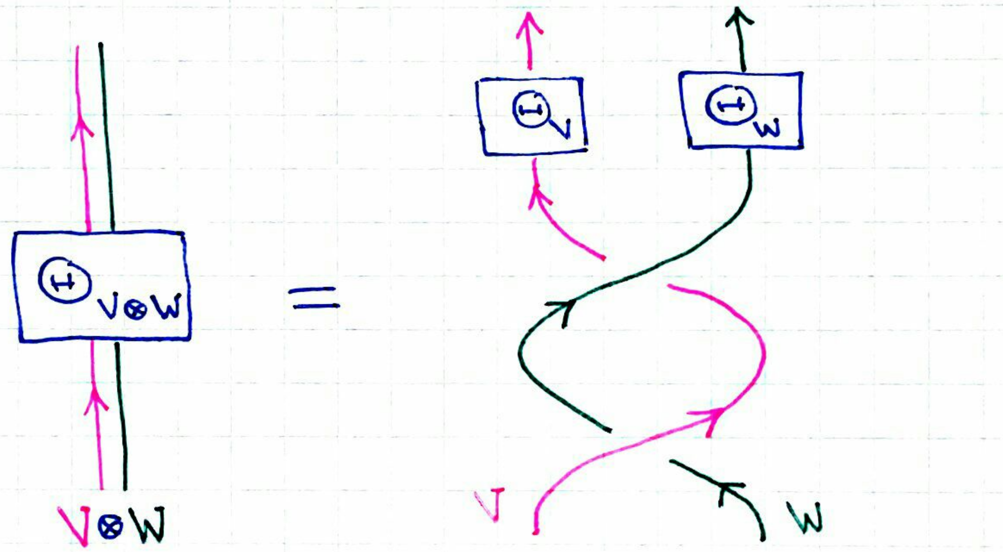
▸ has twists

$$\Theta_{V_\alpha} : V_\alpha \longrightarrow V_\alpha$$

that satisfy

- Θ_{V_α} is H-intertwining isomorphism

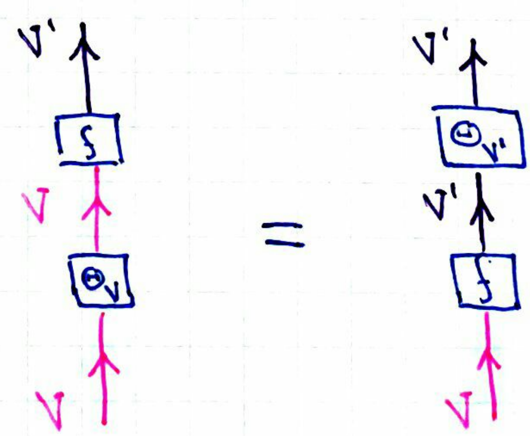
(a): $\Theta_{V\otimes W} = (\Theta_V \otimes \Theta_W) \circ c_{W,V} \circ c_{V,W}$



(b): $\Theta_{V^*} = (\Theta_V)^*$ (transpose)

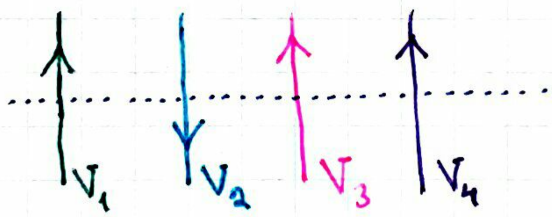
- naturality: whenever $f : V \to V'$ is H-intertwining, we have

$$f \circ \Theta_V = \Theta_{V'} \circ f$$



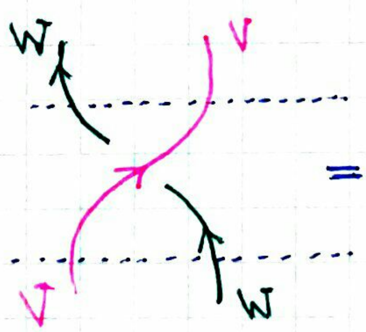
In this setup, to any generic knot diagram of an oriented framed knot colored with one of the representations $V = V_\alpha$ from the collection (more generally: to generic tangle diagram of an oriented framed tangle with each component colored with one of the reps) we can associate a H-intertwining map by reading the diagram from bottom to top and interpreting:

- a generic horizontal line as the tensor product representation of the colors of the strands of the upwards oriented strands intersecting that line and duals of the colors of the downwards oriented strands intersecting that line, in the order they appear along the line, e.g.

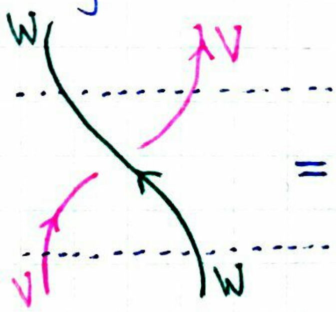


$$= V_1 \otimes V_2^* \otimes V_3 \otimes V_4$$

- a crossing as a braiding or inverse braiding, depending on the type of the crossing, of the representations corresponding to the colors of the crossing strands:



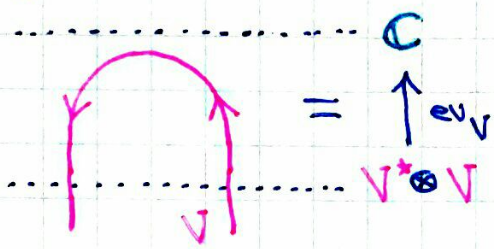
$$= \begin{array}{c} W \otimes V \\ \uparrow c_{V,W} \\ V \otimes W \end{array}$$



$$= \begin{array}{c} W \otimes V \\ \uparrow c_{W,V}^{-1} \\ V \otimes W \end{array}$$

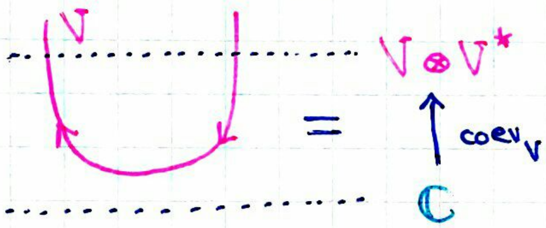
- a left-to-right oriented local maximum with color V as the evaluation map

$$\mathrm{ev}_V : V^* \otimes V \longrightarrow \mathbb{C} \quad , \quad \varphi \otimes v \longmapsto \langle \varphi, v \rangle$$



- a left-to-right oriented local minimum with color V as the coevaluation map

$$\mathrm{coev}_V : \mathbb{C} \longrightarrow V \otimes V^* \cong \mathrm{Hom}(V,V)$$
$$1 \longmapsto \sum e_i \otimes \delta^i \cong \mathrm{id}_V$$



- right-to-left local minima and maxima described soon, after some first remarks

<u>Remarks:</u>

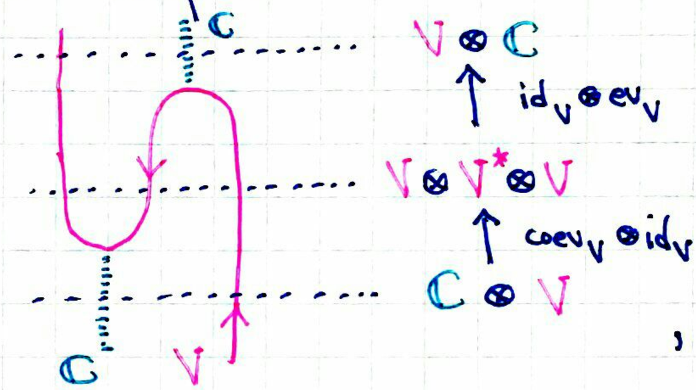
- Braidings $c_{V,W}$ with the required properties exist at least if H has a universal R-matrix $R \in H \otimes H$, but this condition can be slightly relaxed.
- Twists Θ_V with the required properties exist at least if H has a universal R-matrix $R \in H \otimes H$ and a "ribbon element" $\theta \in H$ s.t. $\Delta(\theta) = (R_{21} R)^{-1} (\theta \otimes \theta)$, $\varepsilon(\theta) = 1$ and $\gamma(\theta) = \theta$ — see exercises.
- The evaluation $\mathrm{ev}_V : V^* \otimes V \longrightarrow \mathbb{C}$ and coevaluation $\mathrm{coev}_V : \mathbb{C} \longrightarrow V \otimes V^*$ are indeed H-intertwining (check on your own!)

In order to verify that the H-intertwining map associated to the knot diagram is a knot invariant, we have to check that it is unchanged under the transformations (A), (B), (C), (D), (I'), (II), (III). (actually: oriented, colored versions)

Invariance under (A) and (B) is rather clear from the construction.

Let us then look at (C) for those cases that only have right-to-left oriented extrema. For example, we should compare

with



,

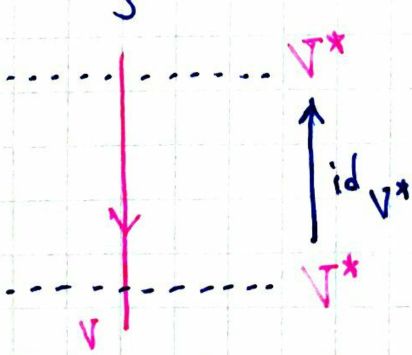
i.e. show the equality $\mathrm{id}_V \overset{?}{=} (\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V)$.

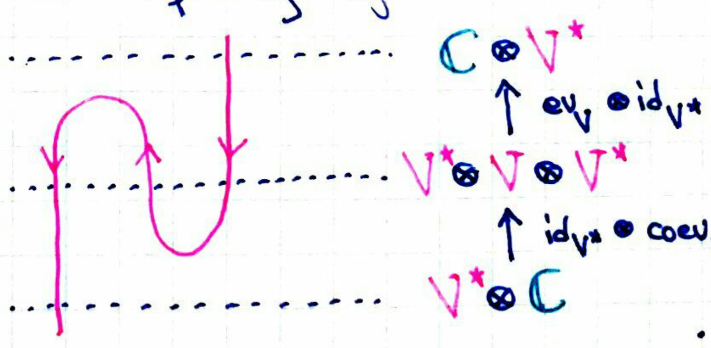
Now this is in fact easy: let $(e_i)_{i=1}^d$ be a basis of V and $(\delta^i)_{i=1}^d$ the dual basis of V^*, and calculate for any $k = 1, \dots, d$:

$$e_k \xmapsto{\mathrm{coev} \otimes \mathrm{id}} \sum_{j=1}^d e_j \otimes \delta^j \otimes e_k \xmapsto{\mathrm{id} \otimes \mathrm{ev}} \sum_{j=1}^d e_j \otimes \langle \delta^j, e_k \rangle$$

$$= \sum_{j=1}^d e_j \cdot \delta_{j,k} = e_k = \mathrm{id}_V(e_k).$$

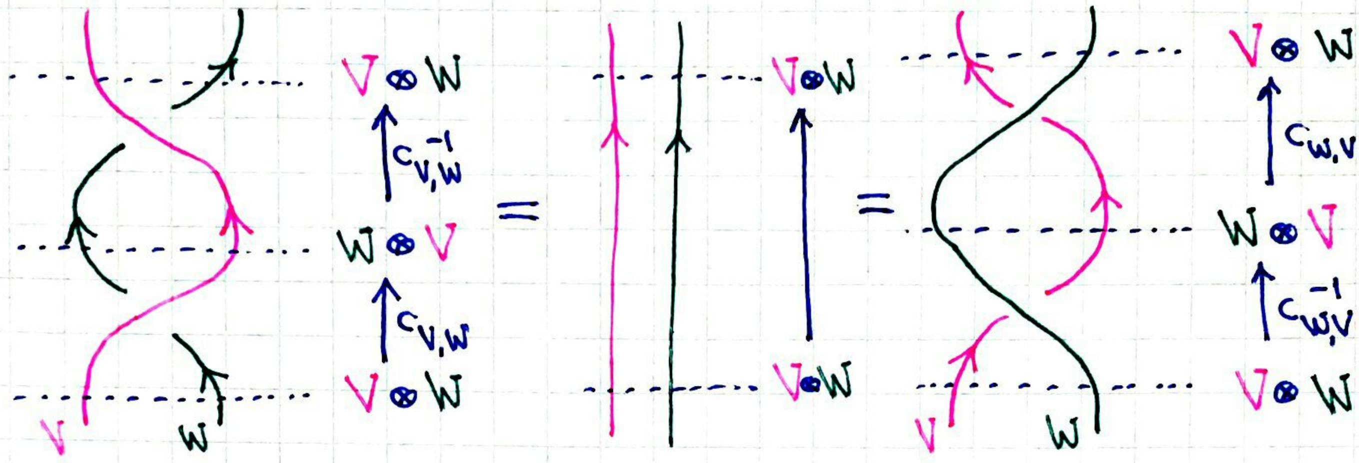
Similarly one shows the equality of





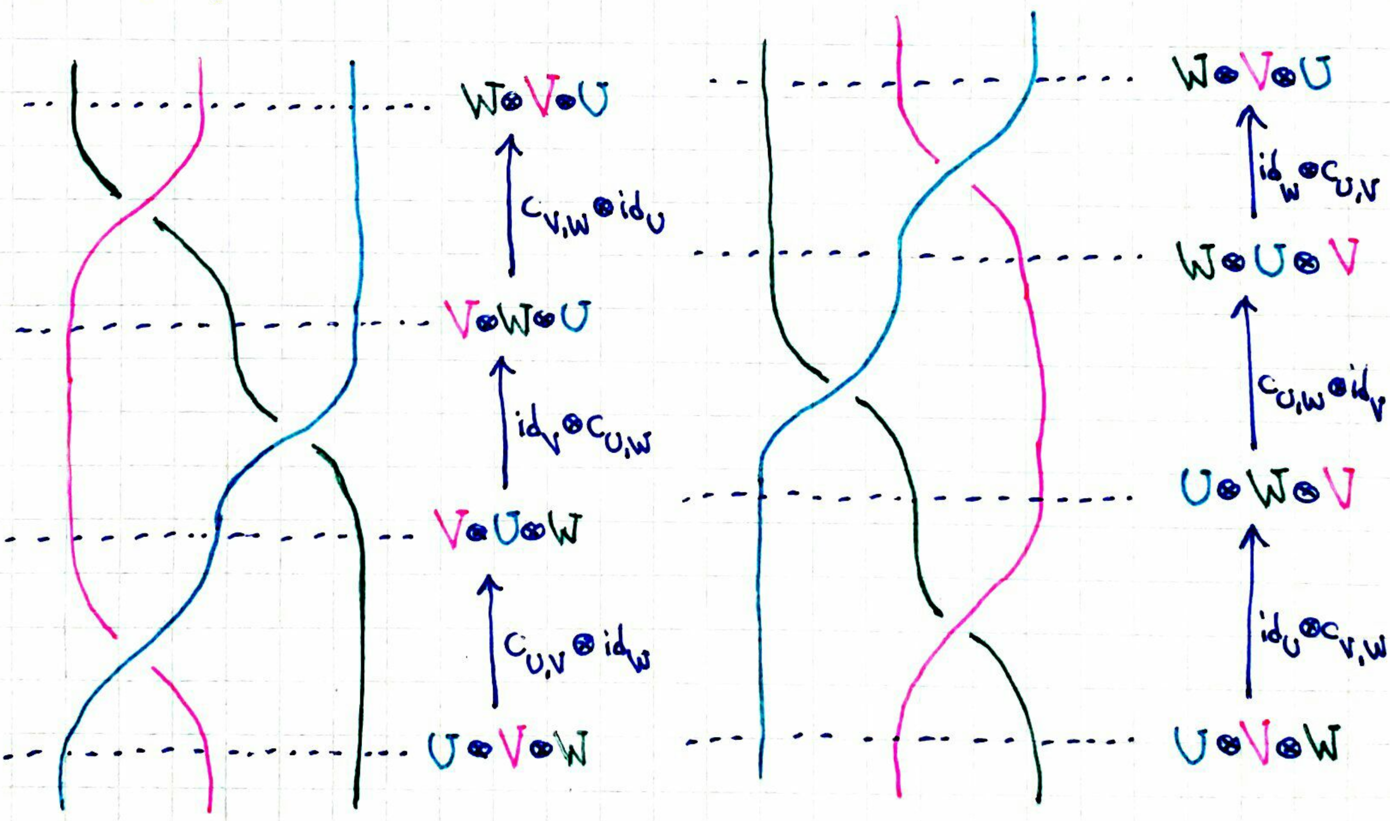
.

Invariance under Reidemeister move (II) is immediate, since to the two types of crossings we assign the braiding and the inverse braiding



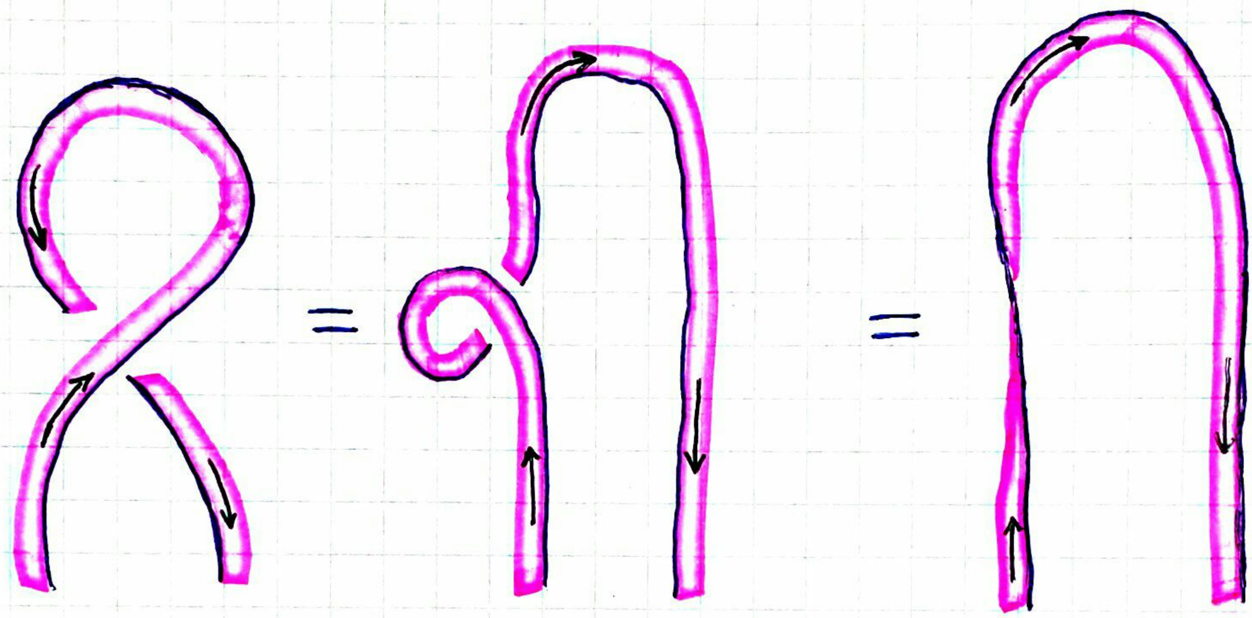
$$c^{-1}_{V,W} \circ c_{V,W} = \mathrm{id}_{V \otimes W} = c_{W,V} \circ c^{-1}_{W,V} .$$

We have also seen that properties of the braiding imply the following invariance under Reidemeister move (III):



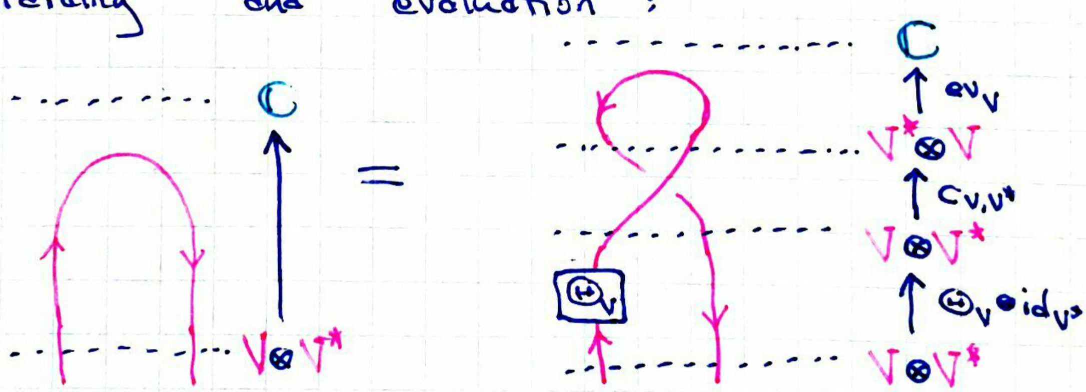
$$(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W})$$

However, we have not yet given the interpretation of left-to-right oriented local maxima and minima as intertwiners. It is here that the framing of the knots/links/tangles is important and the twists Θ_V are employed. Namely, motivated by the following identity of framed oriented strands of "ribbon"

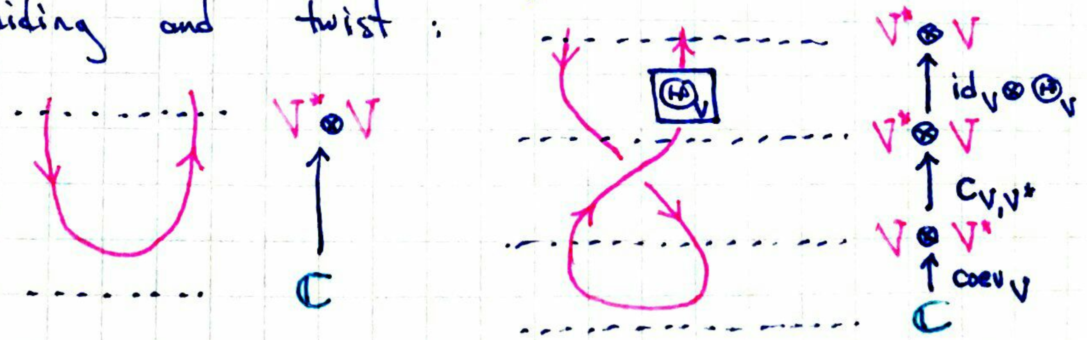


we choose to interpret

- a left-to-right oriented local maximum with color V as a composition of twist, braiding and evaluation:

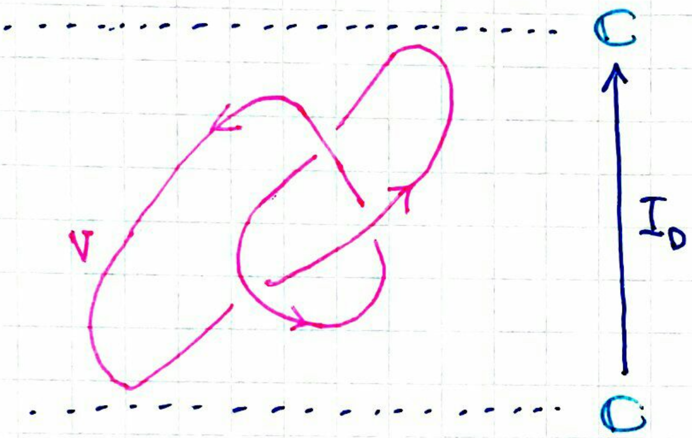


- a left-to-right oriented local minimum with color V as a composition of coevaluation, braiding and twist:



We refer the reader to e.g. [Kassel: "Quantum groups"] for the verification of the invariance under the rest of the transformations of types (A), (B), (C), (D) (I'), (II), (III) for framed oriented colored knots etc.

Then we have associated to any knot diagram D of an oriented framed colored knot (or link) an H-intertwining map $I_D : \mathbb{C} \to \mathbb{C}$



This linear map $\mathbb{C} \to \mathbb{C}$ is a multiplication by some scalar, so we can interpret I_D as a scalar-valued knot invariant. If we do this with e.g. H a quantum group depending on a deformation parameter q, then we get an invariant which is a scalar-valued function of q — in fact typically a polynomial in some fractional power of q. These types of polynomial invariants of knots have turned out very useful.

We briefly outline more concretely a construction of a knot invariant based on $H = \mathcal{U}_q(\mathfrak{sl}_2)$.