Global in time existence of solutions for the heat equation with a superlinear source term

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Introduction

Consider

$$\text{(P)} \qquad \begin{cases} \partial_t u = \Delta u + f(u), & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases}$$

$f \in C^1([0,\infty))$: non-negative, increasing function.

Aim: Global in time existence of sol. of problem (P).

Definition

For a suitable Banach space X (e.g. $X = L^r(\mathbb{R}^N)$ or $L^r_{ul}(\mathbb{R}^N)$),
$u = u(x,t) \in C^{2,1}(\mathbb{R}^N \times (0,T))$ is a *classical sol.* of (P) in X if

- u satisfies the equation pointwisely.
- $\lim_{t\to 0} \|u(\cdot,t) - e^{t\Delta}u_0\|_X = 0$. (not $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_X = 0$)

$(e^{t\Delta}u_0)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\,dy$: sol. of the heat eq.

Existence and Non-existence of solutions

Case $u_0 \in L^\infty(\mathbb{R}^N)$ $\rightsquigarrow$ $\forall u_0 \in L^\infty$, $\exists$ classical sol. of (P) in L^∞.

Case $u_0 \notin L^\infty(\mathbb{R}^N)$ $\rightsquigarrow$ **Singularity** vs. **Nonlinearity**

Weissler '80: Let $u_0 \in L^r(\mathbb{R}^N)$ $(r \geq 1)$, $f(u) = u^p$ $(p > 1)$.
Put $r_c := \frac{N}{2}(p-1)$.

- $r > r_c$ or $r = r_c > 1 \Rightarrow \forall u_0 \in L^r$, $\exists$ classical sol. of (P) in L^r.
- $1 \leq r < r_c \Rightarrow \exists u_0 (\geq 0) \in L^r$ s.t. $\nexists$ nonnegative sol. in L^r.

Critical space

$L^{r_c}(\mathbb{R}^N)$ classifies the existence and nonexistence.

Remark: Let $u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$ for $\lambda > 0$ (Self-similar scaling). If u satisfies $\partial_t u = \Delta u + u^p$, then so does u_λ for all $\lambda > 0$.

$$\|u_\lambda(\cdot,0)\|_{L^r} = \|u_0\|_{L^r} \iff r = \frac{N}{2}(p-1)(= r_c).$$

$\rightsquigarrow$ $L^{r_c}(\mathbb{R}^N)$ is the scale invariant space.

Existence and Non-existence of solutions

F.-Ioku, to appear:

Definition (Uniformly local L^p space, $p \geq 1$)

$$L^p_{ul}(\mathbb{R}^N) := \left\{ u \in L^p_{loc} \,:\, \|u\|_{p,ul} := \sup_{y \in \mathbb{R}^N} \left(\int_{B_1(y)} |u(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L^p(\mathbb{R}^N) \subsetneq \mathcal{L}^p_{ul}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{p,ul}} \subsetneq L^p_{ul}(\mathbb{R}^N) \ (1 \leq p < \infty).$$

Let $F(s) := \displaystyle\int_s^\infty \frac{1}{f(u)}\, du < \infty$ $(s > 0)$ and assume that the limit A exists:

$$A := \lim_{s \to \infty} f'(s)F(s) \implies A \geq 1.$$

Put $r_c(A) := \frac{N}{2} \cdot \frac{1}{A-1}$ $(A > 1)$.

Remark: $f(u) = u^p \Rightarrow f'(s)F(s) = \frac{p}{p-1} (= A)$, $r_c(A) = \frac{N}{2}(p-1)$.

Existence and Non-existence of solutions

Assume $f'(s)F(s) \leq A$ for $s >> 1$.

Existence:

- $(A > 1)$ $r > r_c(A)$ or $r = r_c(A) > 1$
  $\Rightarrow \forall u_0$ with $\dfrac{1}{F(u_0)^{A-1}} \in \mathcal{L}^r_{ul}$, $\exists$ classical sol. of (P) in L^r_{ul}.
- $(A = 1)$ $r \geq \frac{N}{2} \Rightarrow \forall u_0$ with $\dfrac{1}{F(u_0)^r} \in \mathcal{L}^1_{ul}$, $\exists$ sol. of (P) in L^∞.

Nonexistence: $1 \leq r < r_c(A)$ $(A > 1)$ or $0 < r < \frac{N}{2}$ $(A = 1)$
$\Rightarrow \exists u_0 \geq 0$ s.t. $\nexists$ nonnegative sol. of (P) in L^r_{ul} or L^∞.

Remark: Let $u_\lambda(x,t) := F^{-1}[\lambda^{-2}F(u(\lambda x, \lambda^2 t))]$ for $\lambda > 0$ (Quasi scaling). If u satisfies $\partial_t u = \Delta u + f(u)$, then

$$\partial_t u_\lambda = \Delta u_\lambda + f(u_\lambda) + \frac{|\nabla u_\lambda|^2}{f(u_\lambda)F(u_\lambda)}[f'(u)F(u) - f'(u_\lambda)F(u_\lambda)].$$

On the other hand, $\displaystyle\int_{\mathbb{R}^N} \frac{1}{F(u_\lambda(x,0))^{\frac{N}{2}}}\,dx = \int_{\mathbb{R}^N} \frac{1}{F(u_0(x))^{\frac{N}{2}}}\,dx.$

Remarks

- $\dfrac{1}{F(s)^{\frac{N}{2}}} \to \infty$ as $s \to \infty \rightsquigarrow \left\{ \dfrac{1}{F(u_0(x))^{\frac{N}{2}}} = \infty \right\} = \{u_0 = \infty\}$
- Let $A > 1$ and $f'(s)F(s) \le A$ $(s >> 1)$. Then $s \lesssim \frac{1}{F(s)^{A-1}}$.

  This implies that $u_0 \in L^r_{ul}$ proveded that $\dfrac{1}{F(u_0)^{A-1}} \in L^r_{ul}$.

  $\rightsquigarrow \lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^r_{ul}} = 0$ (**expected singularity: L^r_{ul}**)
- $f(u) = u^p \Rightarrow F(s) = \frac{1}{p-1}s^{-(p-1)}$, $f'(s)F(s) = \frac{p}{p-1} = A > 1$.

$$\frac{1}{F(u_0)^{A-1}} \in \mathcal{L}^r_{ul} \iff u_0 \in \mathcal{L}^r_{ul}, \quad r_c(A) = \frac{N}{2}(p-1).$$

- It follows that

$$\begin{cases} f(u) = u^p & \Rightarrow \quad A = \frac{p}{p-1} \to 1 \ \ (p \to \infty), \\ f(u) = e^u & \Rightarrow \quad A = 1, \\ f(u) = e^{u^2} & \Rightarrow \quad A = 1. \end{cases}$$

  Furthermore, $f'(s)F(s) \le 1$ if $f(u) = e^u$ or e^{u^2} $(s >> 1)$.

Main Theorem

Assume $f(0) = 0$ and the existence of $\alpha := \lim_{s\to 0} f'(s)F(s) \;\Rightarrow\; \alpha \geq 1$.

Theorem (Global existence)

Assume that $f'(s)F(s) \leq A$ $(s >> 1)$, $f'(s)F(s) \leq \alpha$ $(0 < s << 1)$, $f'(s)F(s) \leq \max\{\alpha, A\}$ $(s > 0)$, $\alpha < 1 + \frac{N}{2}$. If

$$\int_{\mathbb{R}^N} \frac{1}{F(u_0)^{\frac{N}{2}}}\, dx << 1,$$

then $\exists$ global in time sol. of (P) in $L^{r_c(A)}_{ul}$ $(A > 1)$ or L^∞ $(A = 1)$.

Remark: (1) $\displaystyle\int_{\mathbb{R}^N} \frac{1}{F(u_\lambda(x,0))^{\frac{N}{2}}}\, dx = \int_{\mathbb{R}^N} \frac{1}{F(u_0(x))^{\frac{N}{2}}}\, dx$ $(\lambda > 0)$

(2) $f(u) = u^p \Rightarrow \alpha = \frac{p}{p-1}$.

$$\alpha < 1 + \frac{N}{2} \iff p > 1 + \frac{2}{N} \text{ (Fujita exponent)}$$

Idea of the proof

Local in time existence for the case $A > 1$ and $r = r_c(A)$

We construct a super-solution $\overline{u}$ of (P), that is,

$$\overline{u}(x,t) \geq (e^{t\Delta}u_0)(x) + \int_0^t [e^{(t-s)\Delta} f(\overline{u}(s))](x)\,ds.$$

$\rightsquigarrow \exists$ sol. u of (P) satisfying $0 \leq u(x,t) \leq \overline{u}(x,t)$.

In fact, define $\{u_n\}_{n=0}^{\infty}$ by $u_0(x,t) = (e^{t\Delta}u_0)(x)$ and

$$u_{n+1}(x,t) = (e^{t\Delta}u_0)(x) + \int_0^t [e^{(t-s)\Delta} f(u_n(s))](x)\,ds.$$

Then $0 \leq u_n(x,t) \leq u_{n+1}(x,t) \leq \overline{u}(x,t)$ for all $n = 0, 1, \ldots$.

$\rightsquigarrow u(x,t) := \lim_{n\to\infty} u_n(x,t)$ gives a sol. of (P).

Idea of the proof

Let v be the sol. of

$$\partial_t v = \Delta v + (A-1)v^{\frac{A}{A-1}}, \quad v(0) = \frac{1}{F(u_0)^{A-1}} \in \mathcal{L}_{ul}^{r_c(A)}.$$

Note that $r_c(A) = \frac{N}{2} \cdot \left(\frac{A}{A-1} - 1\right)$. $\rightsquigarrow$ $\exists v$: sol.

Put $\overline{u}(x,t) := F^{-1}(v(x,t)^{-\frac{1}{A-1}})$.

$$\Longrightarrow \partial_t \overline{u} = \Delta \overline{u} + f(\overline{u}) + \frac{|\nabla \overline{u}|^2}{f(\overline{u})F(\overline{u})} \underbrace{[A - f'(\overline{u})F(\overline{u})]}_{\geq 0 \ (\overline{u} >> 1)}.$$

Idea of the proof

Let v be the sol. of

$$\partial_t v = \Delta v + (A-1)v^{\frac{A}{A-1}}, \quad v(0) = \frac{1}{F(u_0)^{A-1}} \in \mathcal{L}_{ul}^{r_c(A)}.$$

$$\uparrow$$

$$\overline{u_0} = \max\{u_0(x), k\} \; (k >> 1)$$

Note that $r_c(A) = \frac{N}{2} \cdot \left(\frac{A}{A-1} - 1\right)$. $\rightsquigarrow$ $\exists v$: sol.

Put $\overline{u}(x,t) := F^{-1}(v(x,t)^{-\frac{1}{A-1}})$. $\rightsquigarrow$ $\overline{u}(x,t) \geq k$

$$\Longrightarrow \; \partial_t \overline{u} = \Delta \overline{u} + f(\overline{u}) + \frac{|\nabla \overline{u}|^2}{f(\overline{u})F(\overline{u})} \underbrace{[A - f'(\overline{u})F(\overline{u})]}_{\geq 0 \; (\overline{u} >> 1)}.$$

$\overline{u}$: supersolution $\rightsquigarrow$ $\exists u$: sol. of (P) s.t. $u(x,t) \leq \overline{u}(x,t)$.

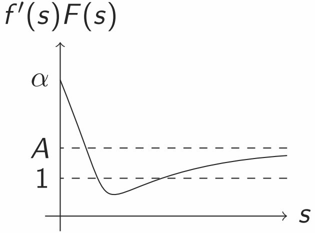
Remark

$$\inf_{x \in \mathbb{R}^N} \overline{u}(x,0) > 0 \quad \Longrightarrow \quad \overline{u} \text{ blows up in finite time}$$

Global existence for the case $\alpha > A > 1$

Assume $\alpha > A$ and $f'(s)F(s) \le \alpha$ for all $s > 0$.

We consider the following equations:



(P) $\quad \partial_t u = \Delta u + f(u),$

(P$_A$) $\quad \partial_t v = \Delta v + (A-1)v^{\frac{A}{A-1}},$

↑ local in time existence

(P$_\alpha$) $\quad \partial_t w = \Delta w + (\alpha-1)w^{\frac{\alpha}{\alpha-1}}.$

1st step: w: sol. of (P$_\alpha$) with $w(0) = \dfrac{1}{F(\overline{u_0})^{\alpha-1}}$. Then

$$\overline{v}(x,t) := w(x,t)^{\frac{A-1}{\alpha-1}} \implies \partial_t \overline{v} \ge \Delta \overline{v} + (A-1)\overline{v}^{\frac{A}{A-1}}.$$

$\rightsquigarrow \exists v$: sol. of (P$_A$) with $v(0) = \dfrac{1}{F(\overline{u_0})^{\alpha-1}}$ s.t. $v \le \overline{v} = w^{\frac{A-1}{\alpha-1}}$.

Global existence for the case $\alpha > A > 1$

2nd step (Local existence):

- $\overline{u}(x,t) := F^{-1}(v(x,t)^{-\frac{1}{A-1}}) \rightsquigarrow \overline{u}$ is a supersolution of (P)
- $\exists u$: sol. of (P) with $u(0) = u_0$ satisfying $u \le \overline{u} \le F^{-1}(w^{-\frac{1}{\alpha-1}})$

3rd step (Global existence):

$$\underline{w}(x,t) := \frac{1}{F(u(x,t))^{\alpha-1}} \implies \partial_t \underline{w} \le \Delta \underline{w} + (\alpha-1)\underline{w}^{\frac{\alpha}{\alpha-1}}.$$

Define $\{W_n\}_{n=0}^{\infty}$ by $W_0(x,t) = \underline{w}(x,t) \le w(x,t)$ and

$$W_{n+1}(x,t) := e^{t\Delta}\left(\frac{1}{F(u_0(x))^{\alpha-1}}\right) + (\alpha-1)\int_0^t e^{(t-s)\Delta} W_n(s)^{\frac{\alpha}{\alpha-1}}\, ds.$$

Then $\underline{w} = F(u)^{-(\alpha-1)} \le W_n \le W_{n+1} \le w$.

$\rightsquigarrow W(x,t) := \lim_{n\to\infty} W_n(x,t)$: sol. of (P_α) s.t. $F(u)^{-(\alpha-1)} \le W$.

$$\iff u(x,t) \le F^{-1}(W(x,t)^{-\frac{1}{\alpha-1}})$$

Global existence for the case $\alpha > A > 1$

Weissler '81

Let $r_c = \frac{N}{2}(p-1) > 1$ and $u_0 \in L^{r_c}(\mathbb{R}^N)$. If $\|u_0\|_{L^{r_c}}$ is sufficiently small, then $\exists$ global solution u of

$$\partial_t u = \Delta u + u^p, \quad u(0) = u_0.$$

Since

$$W(\cdot, 0) = \frac{1}{F(u_0)^{\alpha-1}} \in L^{\frac{N}{2}\cdot\frac{1}{\alpha-1}} = L^{\frac{N}{2}(\frac{\alpha}{\alpha-1}-1)},$$

$$\left\| \frac{1}{F(u_0)^{\alpha-1}} \right\|_{L^{\frac{N}{2}\cdot\frac{1}{\alpha-1}}}^{\frac{N}{2}\cdot\frac{1}{\alpha-1}} = \int_{\mathbb{R}^N} \frac{1}{F(u_0(x))^{\frac{N}{2}}}\, dx: \text{ small},$$

W: sol. of (P_α), exists globally in time.

(P_α) $$\partial_t w = \Delta w + (\alpha-1) w^{\frac{\alpha}{\alpha-1}}$$

Applications

(P1) $$\partial_t u = \Delta u + u^p + u^q, \quad x \in \mathbb{R}^N, \ \ t > 0. \quad (p > q > 1)$$

- $f'(s)F(s) \le \frac{p}{p-1} = \lim_{s\to\infty} f'(s)F(s) = A$ for $s >> 1$.
- $f'(s)F(s) < \frac{q}{q-1} = \lim_{s\to 0} f'(s)F(s) = \alpha$ for all $s > 0$.
- $F(s)^{-\frac{N}{2}} \lesssim s^{\frac{N}{2}(p-1)} + s^{\frac{N}{2}(q-1)}$ for all $s > 0$. $(F(s) = \int_s^\infty \frac{du}{f(u)})$

Corollary 1

Assume $q > 1 + \frac{N}{2}$ ($\iff \alpha < 1 + \frac{N}{2}$). If

$$\int_{\mathbb{R}^N} (|u_0(x)|^{\frac{N}{2}(p-1)} + |u_0(x)|^{\frac{N}{2}(q-1)})\, dx << 1,$$

then $\exists u$: global sol. of (P1) satisfying

$$\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L_{ul}^{\frac{N}{2}(p-1)}} = 0, \quad \lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^{\frac{N}{2}(q-1)}} = 0.$$

Applications

Let $f(u) = e^{u^2}$ and consider

$$\text{(P2)} \qquad \partial_t u = \Delta u + e^{u^2}, \quad x \in \mathbb{R}^N, \ \ t > 0.$$

- $f'(s)F(s) < 1 = \lim_{s\to\infty} f'(s)F(s) = A$ for all $s > 0$.
- $\int_{\mathbb{R}^N} |u_0|^{\frac{N}{2}} e^{\frac{N}{2}|u_0|^2}\, dx < \infty \ \Rightarrow \ \int_{\mathbb{R}^N} \frac{1}{F(u_0(x))^{\frac{N}{2}}}\, dx < \infty$

  (thereshold integrability for local existence)

$\rightsquigarrow$ $\exists u$: local sol. of (P2) s.t. $\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_\infty = 0$ and

$$\lim_{t\to 0} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u(t) - u_0|^{\frac{N}{2}} e^{\frac{N}{2}|u(t)-u_0|^2}\, dx = 0.$$

Remark: Since $f(0) > 0$ and $u_0(x) \geq 0$, u blows up in finite time.

Applications

Let $f(u) = u^\lambda e^{u^2}$ ($\lambda > 1 + \frac{2}{N}$) and consider

$$\text{(P3)} \qquad \partial_t u = \Delta u + u^\lambda e^{u^2}, \quad x \in \mathbb{R}^N, \quad t > 0.$$

- $f'(s)F(s) < 1 = \lim_{s\to\infty} f'(s)F(s) = A$ for $s >> 1$
- $f'(s)F(s) < \frac{\lambda}{\lambda-1} = \lim_{s\to 0} f'(s)F(s) = \alpha < 1 + \frac{N}{2}$ for all $s > 0$.
- $F(s)^{-\frac{N}{2}} \lesssim s^{\frac{N}{2}(\lambda-1)} + s^{\frac{N}{2}(\lambda+1)e^{\frac{N}{2}s^2}}$ for all $s > 0$.

Corollary 2

$$\int_{\mathbb{R}^N} \left(|u_0(x)|^{\frac{N}{2}(\lambda-1)} + |u_0(x)|^{\frac{N}{2}(\lambda+1)} e^{\frac{N}{2}|u_0|^2}\right) dx << 1$$

$\Rightarrow \exists u$: global sol. of (P3) s.t. $\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_\infty = 0$ and

$$\lim_{t\to 0}\left[\left\| |u(t)-u_0|^{\frac{N}{2}(\lambda+1)} e^{\frac{N}{2}|u(t)-u_0|^2}\right\|_{L^1_{ul}} + \|u(t)-u_0\|_{L^{\frac{N}{2}(\lambda-1)}}\right] = 0.$$

Thank you for your kind attention!!