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Multicentric Holomorphic Calculus

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Abstract

We show how for any bounded operator or an element of a Banach algebra one can construct a prcatical power series calculus

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1 Introduction

1.1 Motivation

To start with, consider the following question. Is it possible to have an algorithm, formulated in Banach algebras, such that if an arbitrary element a is given in some algebra $\mathcal{A}$, then the algorithm computes $f(a)$ for a given analytic function f. In this motivation part, let this function be logarithm. We assume that we can create polynomials of a and compute their norms. It turns out that the answer is positive in the following way. We first run an algorithm searching for a polynomial p such that the function f is analytic in a neighborhood of an associated set V_p:

$$V_p = \{\lambda \in \mathbb{C} \mid |p(\lambda)| \leq \|p(a)\|\}.$$

In case of logarithm the requirement reduces to testing

$$0 \notin V_p.$$

Then based on such a polynomial p we proceed to compute a power series representation for the logarithm, which is guaranteed to converge at a. In this paper we collect and discuss properties and computational aspects of these series expansions, while the search algorithm for suitable p has appeared in [11]. We shall now outline the approach in more detail. We should add that in this discussion all computations are assumed to be performed exactly.

1.2 Outline of the approach

Let $a \in \mathcal{A}$ be a given element in a complex unital Banach algebra $\mathcal{A}$ and denote by $\widehat{\sigma(a)}$ the polynomially convex hull of the spectrum of a (that is, fill in the possible holes in $\sigma(a)$). We look at $\widehat{\sigma(a)}$ because if we are given only the element a and assume that our algorithm generates polynomials $p(a)$ (or $p(A)x$ for $x \in X$ when A is an operator in a Banach space X), then all our computations stay in the subalgebra generated by a in $\mathcal{A}$ and we can only try to compute $\widehat{\sigma(a)}$ as it is exactly the spectrum of a in this subalgebra. Let $\Omega \subset \mathbb{C}$ be open and such that $\widehat{\sigma(a)} \subset \Omega$. If f is analytic in Ω, in short, $f \in H(\Omega)$, then the *holomorphic functional calculus* allows one to define $f(a) \in \mathcal{A}$ using the Cauchy integral formula in a way consistent with rational approximation of f.

In the following we omit writing the unit in the algebra, so that for example in the Cauchy formula we write the resolvent as a function from $\mathbb{C} \setminus \widehat{\sigma(a)}$ to $\mathcal{A}$

$$\lambda \mapsto (\lambda - a)^{-1}.$$

Let $\Gamma \subset \Omega$ be a countour surrounding $\widehat{\sigma(a)}$. Then the Cauchy formula defines $f(a)$:

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - a)^{-1} d\lambda. \tag{1.1}$$

However, in order to compute $f(a)$ using (1.1) we need to know the resolvent along Γ. For this we usually have only piecewise smooth approximations which in particular imply that we loose the property of Cauchy integral being path independent. In [11] we outlined a different approach which approximates the resolvent with rational functions. This leads to a simple power series calculus which we approach by first having a look at Taylor series.

Suppose there exists $\lambda_0 \in \mathbb{C}$ such that

$$V := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| \leq \|a - \lambda_0\|\} \subset \Omega.$$

Then we can write for $\lambda \notin V$

$$(\lambda - a)^{-1} = \frac{1}{\lambda - \lambda_0}(1 - \frac{1}{\lambda - \lambda_0}(a - \lambda_0))^{-1} = \sum_{j=0}^{\infty} \frac{(a - \lambda_0)^j}{(\lambda - \lambda_0)^{j+1}} \tag{1.2}$$

and substitute this into (1.1) to get

$$f(a) = \sum_{j=0}^{\infty} \alpha_j (a - \lambda_0)^j \tag{1.3}$$

where the Taylor coefficients of f satisfy

$$\alpha_j = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = r} \frac{f(\lambda)}{(\lambda - \lambda_0)^{j+1}} \, d\lambda.$$

Thus, here the holomorphic functional calculus reduces to substituting a into the variable in the Taylor series expansion of f around the center λ_0.

Our aim in this paper is to discuss the formulas for series expansions in a more general situation where the first degree polynomial $\lambda - \lambda_0$ is replaced by a monic polynomial

$$p(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_d)$$

of degree d with distinct zeros λ_j. As before, let

$$V_p = \{\lambda \in \mathbb{C} \mid |p(\lambda)| \leq \|p(a)\|\}$$

and compute q such that

$$p(\lambda) - p(z) = (\lambda - z)q(\lambda, z).$$

Then clearly the series expansion

$$(\lambda - a)^{-1} = q(\lambda, a) \sum_{j=0}^{\infty} \frac{p(a)^j}{p(\lambda)^{j+1}} \tag{1.4}$$

converges for $\lambda \notin V_p$.

In practical large scale computations with $A \in B(H)$ one can try to run the usual Arnoldi algorithm and draw the corresponding sets V_p and choose a p when V_p is small enough. However, this may not always work. We have recently shown [11] that convergent algorithms do exist. More precisely, there exists a procedure which assumes at each step a finite number of minimization problems involving $\|P(a)\|$ (in case of operators this strictly speaking requires one to work with operator norms and not on $\|P(A)x\|$) over a set of polynomials P of fixed degree to be carried out. It produces a sequence of polynomials p_n and compact sets K_n such that

$$\widehat{\sigma(a)} \subset V_{p_n} \subset K_n \subset K_{n-1}$$

and

$$\widehat{\sigma(a)} = \bigcap_{n \geq 1} K_n.$$

In particular, the representation (1.4) holds for $\lambda \notin V_{p_n}$ with $p = p_n$. So, if $f \in H(\Omega)$ and $\widehat{\sigma(a)} \subset \Omega$ one obtains after a finite number of steps using this procedure a p such that

$$\widehat{\sigma(a)} \subset V_p \subset \Omega.$$

Substituting now (1.4) into (1.1) yields

$$f(a) = \sum_{j=0}^{\infty} c_j(a) p(a)^j \tag{1.5}$$

where c_j's are polynomials with $deg(c_j) < deg(p)$. This is very efficient if $p(a)$ is small. In the extreme case for matrices with minimal polynomial we have $p(A) = 0$ and $f(A) = c_0(A)$, but our emphasis is in the case where $deg(p)$ is relatively small and $\|p(a)\|$ of moderate size.

The expansion (1.5) corresponds to expanding f into a *multicentric power series* or *Jacobi series*

$$f(z) = \sum_{j=0}^{\infty} c_j(z) p(z)^j \tag{1.6}$$

where $\lambda_1, \ldots, \lambda_d$, the roots of p, all have equal importance as local centers. Denote by $\delta_k \in \mathbb{P}_{d-1}$ the Lagrange interpolation basis polynomials at λ_j's

$$\delta_k(\lambda) = \frac{1}{p'(\lambda_k)} \prod_{j \neq k} (\lambda - \lambda_j) \tag{1.7}$$

so that any $P \in \mathbb{P}_{d-1}$ can uniquely be written as

$$P(z) = \sum_{k=1}^{d} P(\lambda_k)\delta_k(z).$$

Let now $c_j \in \mathbb{P}_{d-1}$ be given and put $\alpha_{kj} = c_j(\lambda_k)$. If we now set for $k = 1, \ldots, d$

$$f_k(w) = \sum_{j=0}^{\infty} \alpha_{kj} w^j, \tag{1.8}$$

then we obtain from (1.6) a *multicentric representation* of f w.r.t. centers $\lambda_1, \ldots, \lambda_d$:

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(p(z)). \tag{1.9}$$

We shall first discuss the existence and uniqueness of multicentric representations without smoothness assumptions on f. If, however, f is holomorphic, then each f_k is holomorphic as well and can be expanded into convergent power series (1.8) which can be combined back into a multicentric power series (1.6). It turns out that the Taylor coefficients α_{kj} of f_k can be computed recursively by explicit substitution if the derivatives of f are available at all centers λ_k.

We should point out that we do not assume Ω to be connected. In fact, this allows one to obtain Riesz projectors by defining f to be locally constant. Yet, in multicentric repsentation the functions f_k are in such a case not locally constant.

We shall derive all, as such rather elementary, formulas in detail and provide simple bounds to control the convergence and the error when truncating the representations. We do believe that many of the formulas have been used in the past but we have not found them. Much of the tradition assumes any change of variable to be univalent, here we force it to be d-valent.

We close this introductory section by formulating a result for $f \in H(\Omega)$. Recall that if $\Omega \subset \mathbb{C}$ is open, then we can have compact sets $K_n \subset K_{n+1} \subset \Omega$ so that

$$\Omega = \bigcup_{n \geq 1} K_n.$$

Suppose we are given a compact $K \subset \Omega$. Then by Hilbert Lemniscate Theorem, see e.g. [12], p 158, there exists a polynomial p and a contour Γ surrounding K inside Ω so that

$$\max_{z \in K} |p(z)| < \min_{\lambda \in \Gamma} |p(\lambda)|.$$

This then implies that the corresponding multicentric power series of f converges absolutely inside K.

Remark 1.1. When we work with operators A their powers are often considered either impossible or expensive to be carried out. Observe, however, that if one is able to form $p(A)$ with a relatively low degree polynomial p then the holomorphic functional calculus can be performed on operator-vector level by creating the power series in an obvious manner. Further, even if forming $p(A)$ would be impossible in practise, the calculus can be carried out on vector level, provided that one can compute a reliable estimate of $\|p(A)\|$.

1.3 Remarks on the history

Series expansions of the form (1.6) were introduced and discussed by C.G.J. Jacobi without use of the Cauchy integral. Jacobi died 1851 and the notes, dated 1847, appeared posthumously 1856 [5].

After Jacobi the topic was discussed e.g. by Laguerre and Hilbert before Alfred Kienast discussed it in his inaugural dissertation 1906 [6]. The main properties of Jacobi series are shortly discussed in [14] and [7].

Many results in unit disc for power series have their counterparts for sets bounded by lemniscates if one uses the Jacobi series expansions as basic tool to represent the functions. For example, Fekete [3] has versions on Landau-Schotky theorems and Curtiss [2] studied questions related to boundary behavior.

There exists additionally a notationally appealing formulation of the series as follows. Schweizer [13] defines *Jacobi derivative* by setting

$$\mathcal{D}_p = \sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{\partial}{\partial \lambda_k}.$$

A basically straightforward calculation gives $c_j(z) = \frac{1}{j!}\mathcal{D}_p^j[c_0(z)]$ and thus

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{D}_p^j[c_0(z)] p(z)^j.$$

We shall not discuss this in the following as it seems not to offer advantages towards implementation.

To this end it is natural to wonder why Jacobi series is not part of every day practise. A possible answer could go as follows. A mathematical concept or approach becomes widely known if it is either easy to use as a tool in applications, or it is helpful when one introduces known structures for beginners, or there are still hard unsolved questions nearby. Clearly, without modern computers the computations with Jacobi series based on arbitrary polynomials p would have been really laborious and hence not practical. Second, several naturally emerging generalizations of properties of analytic functions in discs were discussed long ago and that research tradition essentially died out *before* the modern computers were around. Thus, it dropped off from the "toolbox" of basic complex analysis.

My own interest in this grew up as follows. In [8] and in [9] I introduced and studied the polynomial numerical hulls

$$V^k(a) = \bigcap_{deg(p) \le k} V_p$$

and pointed out that intersecting V_p over all polynomials gives the polynomial convex hull of the spectrum. In [10] I studied operator valued meromorphic functions and in particular low rank perturbation theory of operators: what is the inherent invariant property of resolvents which stay invariant in low rank perturbations - while the spectrum may move essentially uncontrolled. It then became natural to ask [11] whether we can compute the spectrum and in particular whether the resolvent could be represented with just one expression allowing the holomorphic functional calculus to be performed with path independent integrations, without need to discretize the integrals. Some authors use Jacobi series to name series expansions by Jacobi polynomials. It took me long time to notice that the series discussed in this paper were also called Jacobi series. Meanwhile I had given lectures on multicentric series - this paper was essentially written before finding the early related papers.

2 Multicentric representation of functions

2.1 Existence and uniqueness

Let $p \in \mathbb{P}_d$ be monic with *distinct zeros* λ_j and denote by $\delta_k \in \mathbb{P}_{d-1}$ the polynomials satisfying

$$\delta_k(\lambda_j) = \delta_{jk}$$

so that

$$\delta_k(\lambda) = \frac{1}{p'(\lambda_k)} \prod_{j \neq k} (\lambda - \lambda_j). \tag{2.1}$$

To discuss the existence and uniqueness of the decomposition

$$f(z) = \sum_{k=1}^{d} \delta_k(z)\ f_k(p(z)) \tag{2.2}$$

we write it as a linear system of equations. Let $w \in \mathbb{C}$ be given and denote by $z_1, \ldots, z_d$ the solutions of

$$p(z) = w. \tag{2.3}$$

As we assume p to have distinct zeros, for small w we have d distinct analytic branches $z_j = z_j(w)$ with $z_j(0) = \lambda_j$. For this discussion we do not need to introduce the full structure of the Riemann surface of $p(z) - w$. It suffices to recall that all branches can be treated as separate analytic functions of a complex variable with the following exception. At *critical points* λ_c we have $p'(\lambda_c) = 0$ and those branches z_j which at the *critical value* $w_c = p(\lambda_c)$ satisfy $z_j(w_c) = \lambda_c$ cannot be treated as analytic functions separately around w_c.

Let now w be a noncritical value, so the roots z_j are all distinct. Denote $f(z_j) = b_j(w)$ and let $b(w) \in \mathbb{C}^d$ be the corresponding vector. Set

$$D(w) = (\delta_k(z_j))_{jk}$$

and denote by $x(w) \in \mathbb{C}^d$ the vector with the unknown function values $f_k(w)$ as components. Then (2.2) can be written as

$$D(w)x(w) = b(w). \tag{2.4}$$

To see that D is nonsingular, put

$$Q(z) = \sum_{k=1}^{d} x_k \delta_k(z)$$

so that $Q \in \mathbb{P}_{d-1}$. Thus (2.4) corresponds to asking solution for the following interpolation problem:

$$Q(z_j) = f(z_j), \text{ for } j = 1, \ldots, d$$

which has a unique solution as the points z_j are assumed to be distinct.

Lemma 2.1. *The function*

$$w \mapsto D(w)^{-1} \tag{2.5}$$

is analytic away from the critical values.

Proof. The matrix valued function $w \mapsto D(w)$ is analytic except at critical values and as it is also nonsingular, its inverse is also analytic. □

Above we needed the values of f at all points z_j in order to pin down the representation. For that purpose we introduce the following concept.

Definition 2.2. *A set $V \subset \mathbb{C}$ is p-balanced if*

$$p^{-1}(p(V)) = V.$$

Thus, V is p-balanced if for every $\lambda \in V$ all roots z_j of

$$p(z) = p(\lambda)$$

also belong to V.

Example 2.3. Sets of the form

$$V = \{\lambda \in \mathbb{C} \mid |p(\lambda)| \le R\}$$

and

$$V_0 = \{\lambda \in \mathbb{C} \mid |p(\lambda)| \le R \text{ and } p(\lambda) \text{ is a noncritical value of } p\}$$

are p-balanced.

We can summarize the situation as follows.

Theorem 2.4. *Let $V_0 \subset \mathbb{C}$ be p-balanced containing no critical points of p. Then for every function*

$$f : V_0 \to \mathbb{C}$$

there exists unique functions

$$f_k : p(V_0) \to \mathbb{C} \tag{2.6}$$

such that for $z \in V_0$

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(p(z)). \tag{2.7}$$

2.2 Continuity of the representation

The multicentric representation inherits all the smoothness of f, as long as we stay away from critical values of p.

Proposition 2.5. *Let $V_0 \subset \mathbb{C}$ be p-balanced containing no critical points of p and suppose f, f_k satisfy (2.7). If $f \in C^m(V_0)$ for some $m \geq 0$, then $f_k \in C^m(p(V_0))$ for all k. If f is analytic, so are f_k.*

Proof. This is clear combining (2.4) and Lemma 2.1. □

It is natural to ask whether one can extend the decomposition smoothly to critical points as well. We shall see later that this is automatic if f is analytic but the following example shows that $f \in C^\infty$ is not enough.

Example 2.6. Let $p(\lambda) = \lambda^2 - 1$ so that we have a critical point at origin. Denoting $\delta_1(z) = (1+z)/2$ and $\delta_2(z) = (1-z)/2$ we obtain

$$f_1(z^2-1) = \frac{1}{2}[f(z) + f(-z)] + \frac{f(z) - f(-z)}{2z} \tag{2.8}$$

$$f_2(z^2-1) = \frac{1}{2}[f(z) + f(-z)] - \frac{f(z) - f(-z)}{2z}. \tag{2.9}$$

As $z \to 0$ we see that $f_k(z^2-1)$ tends to a limit if f is analytic at origin. On the other hand let $f(z) = Re\ z = x$ which is in C^∞, then radial limits exist at origin, however, depending on the angle. In this example we have

$$\delta_1(z) = \frac{1+z}{2}$$

and

$$\delta_2(z) = \frac{1-z}{2}$$

and so we have

$$\sum_{k=1}^{2} \delta_k(z) f_k(z^2-1) = f(z).$$

Critical values come up also when f is defined and analytic near each λ_j but their analytic continuations don't match. Here is an example.

Example 2.7. Consider the Riesz spectral projection which is obtained by assuming f to be identically 1 near one component of the spectrum and vanish in a neighborhood of the rest of the spectrum. We may demonstrate this using Example 2.6.

Let $f \equiv 1$ near 1 and vanish identically near -1. We have, for $|w| < 1$

$$(1+w)^{1/2} = 1 + \frac{1}{2}w - \frac{1}{8}w^2 + \frac{1}{16}w^3 - \cdots$$

and

$$(1+w)^{-1/2} = 1 - \frac{1}{2}w + \frac{3}{8}w^2 - \frac{5}{16}w^3 + \cdots .$$

Let us compute the two-centric representation first around 1. There with $z = (1+w)^{1/2}$

$$\delta_1(z) = 1 + \frac{1}{4}w - \frac{1}{16}w^2 + \frac{1}{32}w^3 - \cdots$$

and

$$\delta_2(z) = -\frac{1}{4}w + \frac{1}{16}w^2 - \frac{1}{32}w^3 - \cdots$$

From (2.8) and (2.9) we obtain

$$f_1(w) = 1 - \frac{1}{4}w + \frac{3}{16}w^2 - \frac{5}{32}w^3 + \cdots$$

$$f_2(w) = \frac{1}{4}w - \frac{3}{16}w^2 + \frac{5}{32}w^3 - \cdots$$

This gives

$$\delta_1(z)f_1(w) = 1 + \frac{1}{16}w^2 - \frac{1}{16}w^3 + \cdots$$

and

$$\delta_2(z)f_2(w) = -\frac{1}{16}w^2 + \frac{1}{16}w^3 - \cdots$$

so their sum is identically 1. Near -1 we have

$$\delta_1(z) = -\frac{1}{4}w + \frac{1}{16}w^2 - \frac{1}{32}w^3 - \cdots$$

and

$$\delta_2(z) = 1 + \frac{1}{4}w - \frac{1}{16}w^2 + \frac{1}{32}w^3 - \cdots$$

which gives

$$\delta_1(z)f_1(w) = -\frac{1}{4}w + \frac{1}{8}w^2 - \frac{3}{32}w^3 + \cdots$$

and

$$\delta_2(z)f_2(w) = \frac{1}{4}w - \frac{1}{8}w^2 + \frac{3}{32}w^3 - \cdots$$

So, near -1 their sum vanishes identically. Suppose now that $A \in B(X)$ is such that with $B = A^2 - 1$ we have $\|B\| < 1$ or more sharply, the spectral radius $\rho(B) < 1$. Since

$$\delta_1(A) = \frac{1+A}{2}, \ \ \delta_2(A) = \frac{1-A}{2}$$

we get using the power series expansions for f_k

$$f(A) = \sum_{k=1}^{2} \delta_k(A)f_k(B) = \frac{1}{2} + A(\frac{1}{2} - \frac{1}{4}B + \frac{3}{16}B^2 - \frac{5}{32}B^3 + \cdots).$$

This satisfies $f(A)^2 = f(A)$ and gives the spectral projection onto the invariant subspace related to the part of spectrum which satisfies $|\lambda^2 - 1| < 1$ and $Re\lambda > 0$.

3 Decomposing the Cauchy integral

3.1 Decomposing the Cauchy kernel

Consider the Cauchy kernel $1/(\lambda - z)$ which gives the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma f(\lambda) \frac{d\lambda}{\lambda - z} \tag{3.1}$$

for analytic functions f. We decompose the kernel and then substituting the decomposition into (3.1) yields the decomposition of all analytic functions f.

The idea is easily seen from factoring

$$p(\lambda) - p(z) = (\lambda - z) q(\lambda, z)$$

and then writing

$$\frac{1}{\lambda - z} = \frac{q(\lambda, z)}{p(\lambda) - p(z)}.$$

Lemma 3.1. *Let*

$$p(\lambda) = \prod_{j=1}^{d} (\lambda - \lambda_j)$$

where all roots λ_k are distinct and let δ_k be as in (2.1). Then

$$p(\lambda) - p(z) = (\lambda - z) \sum_{k=1}^{d} p'(\lambda_k) \delta_k(\lambda) \delta_k(z). \tag{3.2}$$

Proof. Since

$$\delta_k(\lambda) = \frac{1}{p'(\lambda_k)} \prod_{j \neq k} (\lambda - \lambda_j)$$

we have

$$p(\lambda) = (\lambda - \lambda_k) p'(\lambda_k) \delta_k(\lambda). \tag{3.3}$$

Interpolating the constant function gives

$$\sum_{k=1}^{d} \delta_k(z) = 1$$

which together with (3.3) yields

$$p(\lambda) = \sum_{k=1}^{d} (\lambda - \lambda_k) p'(\lambda_k) \delta_k(\lambda) \delta_k(z).$$

Interchanging here λ and z and taking the difference concludes the proof. □

Definition 3.2. *Given a monic $p \in \mathbb{P}_d$ with simple zeros λ_k we set*

$$K_k(\lambda, w) = p'(\lambda_k) \frac{\delta_k(\lambda)}{p(\lambda) - w} = \frac{1}{\lambda - \lambda_k} \frac{p(\lambda)}{p(\lambda) - w}. \tag{3.4}$$

Combining this definition with the previous lemma gives

Proposition 3.3. *For $p(\lambda) \neq p(z)$ we have*

$$\sum_{k=1}^{d} \delta_k(z) K_k(\lambda, p(z)) = \frac{1}{\lambda - z}. \tag{3.5}$$

Remark 3.4. Denote by $z_j = z_j(w)$ the roots of $p(z) = w$, so that

$$\frac{p'(\lambda)}{p(\lambda) - w} = \sum_{j=1}^{d} \frac{1}{\lambda - z_j(w)}.$$

Since

$$p'(\lambda) = \sum_{k=1}^{d} p'(\lambda_k)\delta_k(\lambda)$$

we obtain

$$\sum_{k=1}^{d} K_k(\lambda, w) = \sum_{j=1}^{d} \frac{1}{\lambda - z_j(w)}. \tag{3.6}$$

Example 3.5. Let $p(\lambda) = \lambda^2 - 1$ as in Example 2.6. Then

$$K_1(\lambda, w) = \frac{\lambda + 1}{\lambda^2 - 1 - w} \quad \text{and } K_2(\lambda, w) = \frac{\lambda - 1}{\lambda^2 - 1 - w},$$

while

$$\frac{1}{\lambda - z_1(w)} = \frac{\lambda + \sqrt{1 + w}}{\lambda^2 - 1 - w} \quad \text{and} \quad \frac{1}{\lambda - z_2(w)} = \frac{\lambda - \sqrt{1 + w}}{\lambda^2 - 1 - w}$$

3.2 Decomposing analytic functions

Given a monic polynomial $p \in \mathbb{P}_d$ with simple zeros λ_k and $\rho > 0$, let γ_ρ consist of points $\lambda \in \mathbb{C}$ such that

$$|p(\lambda)| = \rho^d. \tag{3.7}$$

Thus, for small ρ γ_ρ consists of d circular curves around the zeros λ_k, while for large ρ the curve reduces to just one circular contour. The curve γ_ρ divides the plane into one unbounded component, $ext(\gamma_\rho)$ and bounded components, which we denote by $int(\gamma_\rho)$. By maximum principle

$$z \in int(\gamma_\rho) \quad \text{if and only if} \quad |p(z)| < \rho^d. \tag{3.8}$$

Thus we can use (3.5) to decompose our analytic function.

Theorem 3.6. *Let $\Omega \subset \mathbb{C}$ be open containing the zeros λ_k of p. Suppose $\rho > 0$ is such that*

$$int(\gamma_\rho) \cup \gamma_\rho \subset \Omega.$$

Given $f \in H(\Omega)$ set for $|w| < \rho^d$

$$f_k(w) = \frac{1}{2\pi i} \int_{\gamma_\rho} K_k(\lambda, w) f(\lambda) d\lambda. \tag{3.9}$$

Then f_k are analytic in $|w| < \rho^d$ and for $z \in int(\gamma_\rho)$ we have

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(p(z)).$$

Proof. Since $|w| < \rho^d = |p(\lambda)|$ the kernel K_k is analytic in w and therefore so are the functions f_k. The claim then follows from combining the Cauchy integral representation of f with (3.5). $\square$

Remark 3.7. Observe that f_k are defined and analytic also at possible critical values.

We shall consider the convergence of the Taylor series of f_k and the multicentric power series of f below but first we point out that the kernels $K_k(\lambda, w)$ can be used also in defining the decomposition of C^1-functions on the plane.

3.3 Pompeiu's formula

If f is not analytic, but has continuous first derivatives, then $\overline{\partial} f \not\equiv 0$ where $\overline{\partial} f(z) = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) f(x+iy)$. From Stokes's theorem one can derive the following *Pompeiu's formula*.

Lemma 3.8. *Let V be open with boundary consisting of a finite number of continuously differentiable Jordan curves. Suppose u has continuous first derivatives in and up to boundary of V. Then for $z \in V$*

$$u(z) = \frac{1}{2\pi i}\int_{\partial V} \frac{u(\lambda)}{\lambda - z} d\lambda + \frac{1}{2\pi i}\int_V \frac{\overline{\partial} u(\lambda)}{\lambda - z} d\lambda \wedge d\overline{\lambda}. \tag{3.10}$$

Proof. See e.g. p.263 in [4]. $\square$

We may now substitute (3.5) into *Pompeiu's formula* to get the following.

Theorem 3.9. *Let $\Omega \subset \mathbb{C}$ be open containing the zeros λ_k of p. Suppose $\rho > 0$ is such that*

$$int(\gamma_\rho) \cup \gamma_\rho \subset \Omega.$$

Given $f \in C^1(\Omega)$ set for noncritical w such that $|w| < \rho^d$

$$f_k(w) = \frac{1}{2\pi i}\int_{\gamma_\rho} K_k(\lambda, w) f(\lambda) d\lambda + \frac{1}{2\pi i}\int_{int(\gamma_\rho)} K_k(\lambda, w)\, \overline{\partial} f(\lambda)\, d\lambda \wedge d\overline{\lambda} \tag{3.11}$$

Then $f_k \in C^1$ away from critical values and for corresponding $z \in int(\gamma_\rho)$ we have

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(p(z)).$$

Proof. As long as w is noncritical the roots z_j of $p(\lambda) - w$ are simple so that all singularities in the area integral are integrable as

$$K_k(\lambda, w) = p'(\lambda_k)\frac{\delta_k(\lambda)}{\prod_{j=1}^{d}(\lambda - z_j)}.$$

$\square$

4 Computation and estimates for the Jacobi series

4.1 Recursion

We consider next the computation of the multicentric power series assuming that f is analytic and it is possible to compute accurately derivatives of f at roots of p. We show that the coefficients for the power series of f_k

$$f_k(w) = \sum_{n=0}^{\infty} \frac{1}{n!} f_k^{(n)}(0) w^n$$

can be computed using an explicit recursion. The recursion is given in Proposition 4.3, while Proposition 4.5 shows the convergence estimates.

Let us put

$$\varphi_k(z) = f_k(p(z))$$

so that our decomposition reads

$$f = \sum_{k=1}^{d} \delta_k \varphi_k. \tag{4.1}$$

Lemma 4.1.

$$f^{(n)} = \sum_{k=1}^{d} \sum_{m=0}^{n} \binom{n}{m} \delta_k^{(n-m)} \varphi_k^{(m)}. \tag{4.2}$$

Observe that since $\delta_k \in \mathbb{P}_{d-1}$ the polynomials $\delta_k^{(n-m)}$ vanish for $n - m \geq d$.

Lemma 4.2. *If $\psi(z) = g(p(z))$ where $p \in \mathbb{P}_d$ then for $n \geq 1$*

$$\psi^{(n)} = \sum_{m=1}^{n} b_{nm} g^{(m)} \tag{4.3}$$

where the polynomials b_{nm} are determined by

$$b_{n+1,m} = b_{n,m-1}\, p' + b'_{n,m} \tag{4.4}$$

with $b_{n,0} = 0$, $b_{11} = p'$ and $b_{n,m} = 0$ for $m > n$.

For simplicity, we state that $b_{00} = 1$ and observe that $b_{nn} = (p')^n$. We obtain an infinite lower triangular matrix $B = (b_{n,m})$ with polynomial entries:

$$B = \begin{pmatrix} 1 & & & & & \\ & p' & & & & \\ & p^{(2)} & (p')^2 & & & \\ & p^{(3)} & 3p'p^{(2)} & (p')^3 & & \\ & p^{(4)} & 4p'p^{(3)} + 3(p'')^2 & 6(p')^2p'' & (p')^4 & \\ & . & . & . & . & . \end{pmatrix}$$

Proposition 4.3. *We can now solve* $f_j^{(n)}(0)$ *in terms of* $f^{(n)}(\lambda_j)$ *and of* $f_k^{(m)}(0)$ *with* $m = 0, \dots, n-1$ *from*

$$(p'(\lambda_j))^n f_j^{(n)}(0) = f^{(n)}(\lambda_j) - \sum_{k=1}^{d} \sum_{m=0}^{n-1} \binom{n}{m} \delta_k^{(n-m)}(\lambda_j) \sum_{l=0}^{m} b_{ml}(\lambda_j) f_k^{(l)}(0). \quad (4.5)$$

Proof. The formula follows by substituting and noting that $\delta_k(\lambda_j) = \delta_{kj}$.

□

4.2 Convergence estimates

We shall now estimate the convergence of the power series

$$f_k(w) = \sum_{j=0}^{\infty} \alpha_{kj} w^j. \quad (4.6)$$

For $|w| < |p(\lambda)| = \rho^d$ we have from (3.4)

$$K_k(\lambda, w) = \frac{1}{\lambda - \lambda_k} \sum_{j=0}^{\infty} (\frac{w}{p(\lambda)})^j. \quad (4.7)$$

Then we obtain the following representation for the Taylor coefficients of f_k.

Proposition 4.4.

$$\alpha_{kn} = \frac{1}{n!} f_k^{(n)}(0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\lambda)}{p(\lambda)^n} \frac{d\lambda}{\lambda - \lambda_k}. \quad (4.8)$$

This gives us immediately simple bounds. In fact, let

$$M(\rho) = M(\rho, p, f) = \max_{|p(\lambda)| \le \rho^d} |f(\lambda)| \quad (4.9)$$

and put

$$L_k(\rho) = \frac{1}{2\pi} \int_{\gamma_\rho} \frac{|d\lambda|}{|\lambda - \lambda_k|}. \quad (4.10)$$

Then clearly

Proposition 4.5.

$$\frac{1}{n!} |f_k^{(n)}(0)| \le L_k(\rho) \, M(\rho) \, \rho^{-dn} \quad (4.11)$$

and

$$f_k(w) = \sum_{n=0}^{\infty} \frac{1}{n!} f_k^{(n)}(0) \, w^n \quad (4.12)$$

holds for $|w| < \rho^d$ *with*

$$|f_k(w)| \le L_k(\rho) \, M(\rho) \, \frac{\rho^d}{\rho^d - |w|}. \quad (4.13)$$

Remark 4.6. This bound (4.13) can be obtained directly without power series representation by estimating from (3.4) along γ_ρ

$$|K_k(\lambda, w)| \leq \frac{1}{|\lambda - \lambda_k|} \frac{\rho^d}{\rho^d - |w|}.$$

Then (4.13) follows from

$$|f_k(w)| \leq \frac{1}{2\pi} \int_{\gamma_\rho} |K_k(\lambda, w)| \, |f(\lambda)||d\lambda|.$$

4.3 Further estimates

Consider next the multicentric power series

$$f(z) = \sum_{j=0}^{\infty} c_j(z) p(z)^j \tag{4.14}$$

where $c_j \in \mathbb{P}_{d-1}$. The convergence of this series is directly linked with those of f_k's. In fact, from

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(p(z)$$

we obtain

$$c_j(z) = \sum_{k=1}^{d} \delta_k(z) \alpha_{kj} \tag{4.15}$$

which gives substituting $\lambda_k = z$

$$\alpha_{kj} = c_j(\lambda_k). \tag{4.16}$$

Therefore, bounds for the Taylor coefficients α_{kj} of f_k's and bounds for the convergence of (4.14) carry essentially the same information.

As before, suppose $f \in H(\Omega)$ and $int(\gamma_\rho) \cup \gamma_\rho \subset \Omega$. Then (4.14) converges for $|p(z)| < \rho^d$ and for $z \in int(\gamma_\rho)$

$$c_j(z) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{q(\lambda, z) f(\lambda)}{p(\lambda)^{j+1}} d\lambda. \tag{4.17}$$

Here

$$q(\lambda, z) = \frac{p(\lambda) - p(z)}{\lambda - z}.$$

This was the starting point for our study, see introduction above and [ON, 09]. In order to estimate c_j directly from this we put

$$C(\rho) = \frac{1}{2\pi} \int_{\gamma_\rho} \frac{|q(\lambda, z)|}{|p(\lambda)|} |d\lambda| \tag{4.18}$$

We formulate for reference the following simple fact.

Lemma 4.7. *For $p \in \mathbb{P}_d$ there exist constants D_1 and D_2 such that for all $\rho > 0$, $z \in int(\gamma_\rho)$ and $k = 1, \dots, d$*

$$|\delta_k(z)| \leq D_1 + D_2\, \rho^{d-1} \tag{4.19}$$

Then in particular

$$|c_j(z)| \leq (D_1 + D_2\, \rho^{d-1}) \sum_{k=1}^{d} |\alpha_{kj}|. \tag{4.20}$$

Proof. For large ρ the curve γ_ρ is essentially a circle of radius ρ and as δ_k is of degree $d-1$ the claim follows.

□

Next we focus on $C(\rho)$.

Proposition 4.8. *Let $p \in \mathbb{P}_d$ be monic with distinct zeros. Then there exists C such that for all $\rho > 0$*

$$C(\rho) = \frac{1}{2\pi} \int_{\gamma_\rho} \frac{|q(\lambda, z)|}{|p(\lambda)|} |d\lambda| \leq C. \tag{4.21}$$

In particular for $z \in int(\gamma_\rho)$

$$|c_j(z)| \leq CM(\rho)\rho^{-dj}. \tag{4.22}$$

Proof. From (4.17) we obtain, as along γ_ρ we have $|p(\lambda)| = \rho^d$

$$|c_j(z)| \leq C(\rho)M(\rho)\rho^{-dj}.$$

Thus, we need to conclude that $C(\rho)$ is uniformly bounded. It is clear that $C(\rho)$ is continuous. Consider first small values of ρ. Since

$$q(\lambda, z) = \sum_{k=1}^{d} p'(\lambda_k)\delta_k(\lambda)\delta_k(z) = p(\lambda) \sum_{k=1}^{d} \frac{\delta_k(z)}{\lambda - \lambda_k}, \tag{4.23}$$

we obtain by the previous estimate

$$C(\rho) \leq (D_1 + D_2\rho^{d-1}) \sum_{k=1}^{d} L_k(\rho).$$

This shows that $C(\rho)$ is bounded for small ρ, as we show below that $L_k(\rho)$ is uniformly bounded in ρ.

Consider now large values of ρ. However, it follows from the following Lemma that

$$\limsup_{\rho \to \infty} C(\rho) \leq d,$$

which completes the proof. □

Lemma 4.9.

$$\max_{z, \lambda \in \gamma_\rho} \frac{|q(\lambda, z)|}{|p(\lambda)|} \leq d\, \rho^{-1} + \mathcal{O}(\rho^{-2}) \tag{4.24}$$

as $\rho \to \infty$.

Proof. Let

$$p(\lambda) = \lambda^d + a_1\lambda^{d-1} + \dots$$

and assume ρ is large enough so that γ_ρ is nearly circular. Along γ_ρ we have $p(\lambda(\varphi)) = \rho^d e^{id\varphi}$, so that with some constant b

$$\lambda(\varphi) = \rho e^{i\varphi} - a_1 + b\rho^{-1}e^{-i\varphi} + \mathcal{O}(\rho^{-2}) \tag{4.25}$$

and in particular

$$|\lambda(\varphi)| = \rho + \mathcal{O}(1). \tag{4.26}$$

In order to estimate $|q(\lambda, z)|$ we write

$$q(\lambda, z) = \frac{p(\lambda) - p(z)}{\lambda - z}.$$

Since along γ_ρ

$$\frac{|\lambda^j - z^j|}{|\lambda - z|} \le |\lambda|^{j-1} + |\lambda|^{j-2}|z| + \dots + |z|^{j-1} \le j(\rho + \mathcal{O}(1))^{j-1}$$

we obtain

$$|q(\lambda, z)| \le d\,(\rho + \mathcal{O}(1))^{d-1} + (d-1)|a_1|(\rho + \mathcal{O}(1))^{d-2} + \dots.$$

Thus

$$\frac{|q(\lambda, z)|}{|p(\lambda)|} = \frac{|q(\lambda, z)|}{\rho^d} \le d\,\rho^{-1} + \mathcal{O}(\rho^{-2})$$

completing the proof of the lemma.

□

It remains to consider the constants $L_k(\rho)$.

Proposition 4.10. *Let $p \in \mathbb{P}_d$ be monic with distinct zeros. Then there exists a constant L such that for all $k = 1, \dots, d$ and $\rho > 0$*

$$L_k(\rho) = \frac{1}{2\pi}\int_{\gamma_\rho} \frac{|d\lambda|}{|\lambda - \lambda_k|} \le L. \tag{4.27}$$

Proof. Again, for ρ small enough γ_ρ consists of d nearly circular contours and it is easily concluded that integral around λ_k approaches 1 as ρ decreases while the other integrals tend to 0. On the other hand for ρ large enough γ_ρ consists just one nearly circular component and the integral around it again approaches 1 as $\rho \to \infty$. Since $L_k(\rho)$ is continuous in ρ it is uniformly bounded. □

Remark 4.11. It would be interesting to know how large the constants C and L can be. For example, if $p(\lambda) = \lambda^d - 1$, then for $\rho = 1$ γ_ρ surrounds each root of unity and visits in between at origin.

4.4 Polynomial approximations

Given

$$f(z) = \sum_{j=0}^{\infty} c_j(z)p(z)^j$$

denote

$$F_n(z) = \sum_{j=0}^{n} c_j(z)p(z)^j \tag{4.28}$$

and put for the error along γ_ρ

$$E_n(\rho) = \max_{z\in\gamma_\rho} |f(z) - F_n(z)|. \tag{4.29}$$

We have the following bound for the error.

Proposition 4.12. *Suppose f is analytic on and inside γ_ρ and let $r < \rho$. Then*

$$E_n(r) \leq C(\rho)M(\rho)\ \frac{\rho^d}{\rho^d - r^d}\ (\frac{r}{\rho})^{(n+1)d}. \tag{4.30}$$

The following can be viewed as an adaptation of Schwarz lemma.

Proposition 4.13. *Let $\Omega \subset \mathbb{C}$ be a domain and assume that $Z(\rho) \subset \Omega$. Then for $r < \rho$*

$$E_n(r) \leq E_n(\rho)\ \ (\frac{r}{\rho})^{(n+1)d}. \tag{4.31}$$

Proof. Along $\partial Z(\rho)$ we have

$$|\frac{f(z) - F_n(z)}{p(z)^{n+1}}| \leq E_n(\rho)\ \rho^{-d(n+1)} \tag{4.32}$$

which by maximum principle holds also for $z \in Z(\rho)$. □

Next we consider Rouché's theorem.

Proposition 4.14. *If for some $r < \rho$ we have along γ_r*

$$|F_n(z)| < |f(z)| \tag{4.33}$$

then f has at least $d(n+1)$ zeros inside γ_r.

Proof. By Rouché's theorem f and $f - F_n$ have equally many zeros, counted with multiplicities, inside γ_r. But $f - F_n$ vanishes there at least $d(n+1)$ times.

□

5 Basic operations

5.1 Evaluation

Suppose that the multiplication in $\mathcal{A}$ is the costly operation compared to forming sums and scalar multiplications. Given $a \in \mathcal{A}$ and a holomorphic f consider evaluation of $f(a)$ using approximations $F_n(a)$ as in Section 4.4. We have organized the recursion above to first compute the coefficients α_{kj} after which one can compute either the series $f_k(p(a))$ and then take the combination using $\delta_k(a)$ or one proceeds to compute the coefficient polynomials

$$c_j(a) = \sum_{k=1}^{d} \alpha_{kj}\delta_k(a)$$

and then forms $F_n(a)$ directly. If n is large compared with d then clearly the former contains fewer multiplications in the algebra while the latter is to be preferred if d is large compared with n. Further, if n is not chosen in advance, each new power of $p(a)$ requires d+1 multiplications in the former and only two in the latter. Thus the best organization may vary from case to case. One should also notice that the different orders of summations may have different error sensitivities.

If the algebra is just the complex number field, then more can be said. In Lagrange interpolation *barycentric* interpolation formula is often preferred. A discussion about this can be found in [1]. In our notation Lagrange interpolation corresponds to replacing f by

$$c_0(z) = \sum_{k=1}^{d} \delta_k(z) f_k(0)$$

We arrive at barycentric form as follows. Since

$$1 = \sum_{k=1}^{d} \delta_k(z) = p(z) \sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{1}{z-\lambda_k},$$

and

$$c_0(z) = \sum_{k=1}^{d} \delta_k(z) f_k(0) = p(z) \sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{f_k(0)}{z-\lambda_k},$$

we obtain by dividing

$$c_0(z) = \frac{\sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{f_k(0)}{z-\lambda_k}}{\sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{1}{z-\lambda_k}}.$$

Formally this would suggest a generalization as follows:

Compute $w = p(z)$ from

$$w^{-1} = \sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{1}{z-\lambda_k}$$

and evaluate

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w)$$

from

$$f(z) = w \sum_{k=1}^{d} \frac{1}{p'(\lambda_k)} \frac{f_k(w)}{z - \lambda_k}.$$

It is clear that we cannot effectively use this formulation in functional calculus in Banach algebras, as we do not have the necessary inverses at hand.

5.2 Adding and multiplying

Suppose $f(z) = \sum_1^d \delta_k(z) f_k(p(z))$ and $g(z) = \sum_1^d \delta_k(z) g_k(p(z))$. Then the sum can be formed in an obvious way:

$$f(z) + g(z) = \sum_{k=1}^{d} \delta_k(z)(f_k(p(z)) + g_k(p(z))). \tag{5.1}$$

To obtain a formula for the product of two functions one has to have the polynomials $\delta_m \delta_n \in \mathbb{P}_{2d-2}$ precomputed in a suitable way. Given any $Q \in \mathbb{P}_{2d-2}$ there are unique polynomials $R, S \in \mathbb{P}_{d-1}$ such that

$$Q = R + S\, p.$$

Using this one computes for $1 \leq i, j \leq d$ polynomials

$$R_{ij}(z) = \sum_{k=1}^{d} r_{ijk}\ \delta_k(z) \text{ and } S_{ij}(z) = \sum_{k=1}^{d} s_{ijk}\ \delta_k(z)$$

such that

$$\delta_i(z)\delta_j(z) = R_{ij}(z) + p(z)S_{ij}(z). \tag{5.2}$$

Then denoting

$$h_k(w) = \sum_{i,j=1}^{d} (r_{ijk} + s_{ijk}\ w)\ f_i(w)\ g_j(w) \tag{5.3}$$

we obtain

$$f(z)g(z) = \sum_{k=1}^{d} \delta_k(z) h_k(p(z)). \tag{5.4}$$

Observe that (5.2) depends only on the polynomial p.

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