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AMS subject classifications: 30C65, 46T05

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CONVEXITY PROPERTIES OF QUASIHYPERBOLIC BALLS ON BANACH SPACES

ANTTI RASILA AND JARNO TALPONEN

Abstract. We study convexity and starlikeness of metric balls on Banach spaces when the metric is the quasihyperbolic metric or the distance ratio metric. In particular, problems related to these metrics on convex domains, and on punctured Banach spaces, are considered.

1. Introduction

In this paper, we deal with Banach manifolds, which are obtained by defining a conformal metric on non-trivial subdomains of a given Banach space. An example of such metric is the quasihyperbolic metric on a domain of a Banach space. It is obtained from the norm-induced metric by adding a weight, which depends only on distance to the boundary of the domain. The quasihyperbolic metric of domains in $\mathbb{R}^n$ was first studied by F.W. Gehring and his students B. Palka [4] and B. Osgood [3] in 1970's. It has turned out to be a useful tool in, e.g., the theory of quasiconformal mappings. In particular, quasihyperbolic metric plays a crucial role in the theory of quasiconformal mappings in Banach spaces, developed by J. Väisälä in the series of articles [10, 11, 12, 13, 14]. This is due to the fact that many of the tools used in the Euclidean space are not available in the infinite-dimensional setting (see [14]).

We mainly study the question of how the geometry of the Banach space norm translates into the properties of the quasihyperbolic metric. In particular, we consider convexity and starlikeness of quasihyperbolic balls in the punctured Banach space $\Omega = \mathrm{X} \setminus \{0\}$. This problem was posed in $\mathbb{R}^n$ by M. Vuorinen [19], and studied by R. Klén in [5, 6] and J. Väisälä in [16]. Many of the techniques used there are specific to $\mathbb{R}^n$. In the general Banach space setting a very different approach is required.

Our main results are the following. In Theorem 3.1 we show that each ball in the distance ratio metric (the j-metric) defined on a proper subdomain of a Banach space is starlike for radii $r \leq \log 2$, partly generalizing a result of Klén [6, Theorem 3.1]. In Theorem 4.1, which is an improvement of a result of O. Martio and J. Väisälä [9, 2.13],

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we show that the j-balls and the quasihyperbolic balls defined on a convex domain of a Banach space are convex. Then, in Theorem 5.2, we show that there exists a constant $R > 0$ such that all j-balls with radius $r \leq R$ are convex in the punctured Banach spaces, under certain usual assumptions related to the local geometry. We also give a counterexample, which settles a question posed by O. Martio and J. Väisälä [9, 2.14] concerning quasihyperbolic geodesics on uniformly convex Banach spaces. Related problems involving quasihyperbolic geodesics have been studied in $\mathbb{R}^n$ by G.J. Martin [7] in the 1980's, and several authors thereafter. Finally, in Theorem 5.7, we consider convexity of quasihyperbolic balls on punctured Banach spaces.

2. Preliminaries

First we recall a few basic results and definitions. Unless otherwise stated, we will assume that X is a Banach space with $\dim \mathrm{X} \geq 2$, and that $\Omega \subsetneq \mathrm{X}$ is a domain. Open and closed balls in X are

$$\mathbf{U}(x,r) := \mathbf{U}_{\|\cdot\|}(x,r) := \{y \in \mathrm{X} : \|x-y\| < r\}$$

$\mathbf{B}(x,r) := \mathbf{B}_{\|\cdot\|}(x,r) := \{y \in \mathrm{X} : \|x-y\| \leq r\}$, and $\mathbf{S}(x,r) := \partial\mathbf{B}(x,r)$. A set $\Omega \subset \mathrm{X}$ is called *convex* if the line segment

$$[x,y] := \{tx + (1-t)y : t \in [0,1]\} \subset \Omega \text{ for all } x, y \in \Omega,$$

and *starlike* with respect to $x_0 \in \Omega$ if

$$[x_0,y] := \{tx_0 + (1-t)y : t \in [0,1]\} \subset \Omega \text{ for all } y \in \Omega.$$

Observe that the use of notation $[x,y]$ here is different from some texts in Banach spaces. Obviously a set Ω is convex if and only if it is starlike with respect to every point $x_0 \in \Omega$.

2.1. **Paths and line integrals.** In what follows a *path* in a metric space (X, d) is a continuous mapping γ of the unit interval $I = [0,1]$ into X. If $J = [a,b] \subset I$ is a closed subinterval, then the *length* of a path $\gamma\colon I \to \mathrm{X}$ restricted to J is

$$\ell_d(\gamma,a,b) = \sup \sum_{i=1}^{n} d\big(\gamma(t_i), \gamma(t_{i+1})\big), \tag{2.1}$$

where the supremum is taken over all sequences $a = t_1 \leq t_2 \leq \ldots \leq t_n \leq t_{n+1} = b$. The (total) length of γ is $\ell_d(\gamma) = \ell_d(\gamma,0,1)$. A path γ is *rectifiable* if its length is finite.

Given a rectifiable path $\gamma\colon I \to \mathrm{X}$ such that $\ell_d(\gamma,0,s)$ is absolutely continuous with respect to s, we denote the *length element* of γ by

$$\|D\gamma\| = \|D\gamma(s)\| = \frac{d}{ds}\ell(\gamma,0,s) \quad \text{for a.e. } s \in I. \tag{2.2}$$

Recall that an increasing absolutely continuous function is a.e. differentiable and can be recovered by integrating its derivative. Thus

$$\ell(\gamma,0,t)=\int_0^t \|D\gamma\|\ ds=\int_0^t \|d\gamma\|,$$

where the last integral can be interpreted as the Stieltjes integral with respect to integrator $\ell_d(\gamma,0,t)$, or equivalently, the Lebesgue integral, under the formal convention that

$$\|d\gamma\|=\|D\gamma\|\ ds.$$

In this paper both interpretations for the integrals are useful. Note that for instance the parameterization with respect to the arc length is absolutely continuous. Obviously, any rectifiable path can be approximated uniformly by an absolutely continuous path, e.g., a broken line. If γ is a path in a Banach space X, we will denote its Gâteaux derivative by

$$D\gamma(t):=\lim_{h\to 0}\frac{\gamma(t+h)-\gamma(t)}{h},$$

provided that it exists. Observe that if γ is Gâteaux differentiable at t, then $\|D\gamma(t)\|=\|(D\gamma(t))\|$. We note that differentiation of Banach space valued functions can also be studied by means of the Bochner integral. This approach is effective especially in Banach spaces with the so-called Radon-Nikodým property (RNP), which means that any absolutely continuous path starting from the origin can be recovered by Bochner integrating its Gâteaux derivative. For basic information about these concepts we refer to [1], see also [2].

2.2. **Quasihyperbolic metric.** Let X be a Banach space with $\dim \mathrm{X}\geq 2$, and suppose that $\Omega\subsetneq \mathrm{X}$ is a domain. For $x\in\Omega$, let $d(x)$ denote the distance $d(x,\partial\Omega)$. We define the *quasihyperbolic length* of γ by

$$\ell_k(\gamma):=\int_\gamma \frac{\|dx\|}{d(x)}$$

then the *quasihyperbolic distance* of points $x,y\in\Omega$ is the number

$$k_\Omega(x,y):=\inf_\gamma \ell_k(\gamma)$$

where the infimum is taken over all rectifiable arcs γ joining x and y in Ω. Quasihyperbolic balls are

$$\begin{aligned}\mathbf{U}_k(x,r) &:= \{y\in\Omega : k_\Omega(x,y)<r\},\\ \mathbf{B}_k(x,r) &:= \{y\in\Omega : k_\Omega(x,y)\leq r\}.\end{aligned}$$

It is well known [3, Lemma 1] that in the finite-dimensional case there is a quasihyperbolic geodesic between any two points. By [15, Theorem 2.5], for a reflexive Banach space X and a convex subdomain $\Omega\subsetneq \mathrm{X}$ there always exists a quasihyperbolic geodesic connecting $x,y\in\Omega$. One of the peculiarities of this topic is that it is not known whether this holds for general Banach spaces (see also [15, Section 6]). It is easy to check that multiplication by a constant $C \neq 0$ is a quasihyperbolic isometry on $\Omega = \mathrm{X} \setminus \{0\}$.

2.3. **Distance-ratio metric.** The quasihyperbolic distance is often difficult to compute in practice. For this reason, we consider another related quantity, the distance-ratio metric. This metric was originally introduced by Gehring and Palka in [4]. We use a version that is due to Vuorinen [17]. Let X be a Banach space with $\dim \mathrm{X} \geq 2$, and suppose that $\Omega \subsetneq \mathrm{X}$ is a domain. Write

$$a \vee b := \max\{a, b\}, \qquad a \wedge b := \min\{a, b\}.$$

The *distance-ratio metric*, or *j-metric*, on Ω is defined by

$$j_\Omega(x, y) := \log\left(1 + \frac{\|x - y\|}{d(x) \wedge d(y)}\right), \qquad x, y \in \Omega. \tag{2.3}$$

Again, the balls with respect to the j-metric are

$$\begin{aligned} \mathbf{U}_j(x, r) &:= \{y \in \Omega : j_\Omega(x, y) < r\}, \\ \mathbf{B}_j(x, r) &:= \{y \in \Omega : j_\Omega(x, y) \leq r\}. \end{aligned}$$

It is well known that the norm metric, the quasihyperbolic metric and the distance-ratio metric define the same topology on Ω. It is well known that the topologies on Ω induced by the norm, the j-metric and the k-metric coincide. In fact, the j-metric is an inner metric of the quasihyperbolic metric.

2.4. **Geometric control of Banach spaces.** Next we will recall for convenience two essential moduli related to the geometry of Banach spaces. The *modulus of convexity* $\delta_{\mathrm{X}}(\epsilon)$, $0 < \epsilon \leq 2$, is defined by

$$\delta_{\mathrm{X}}(\epsilon) := \inf\{1 - \|x + y\|/2 : \ x, y \in \mathrm{X},\ \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon\},$$

and the *modulus of smoothness* $\rho_{\mathrm{X}}(\tau)$, $t > 0$ is defined by

$$\rho_{\mathrm{X}}(\tau) := \sup\{(\|x + y\| + \|x - y\|)/2 - 1,\ x, y \in \mathrm{X},\ \|x\| = 1,\ \|y\| = \tau\}.$$

The Banach space X is called *uniformly convex* if $\delta_{\mathrm{X}}(\epsilon) > 0$ for all $\epsilon > 0$, and *uniformly smooth* if

$$\lim_{\tau \to 0^+} \frac{\rho_{\mathrm{X}}(\tau)}{\tau} = 0.$$

Moreover, a space X is uniformly convex (resp. uniformly smooth) of power type $p \in [1, \infty)$ if $\delta_{\mathrm{X}}(\epsilon) \geq K\epsilon^p$ (resp. $K\tau^p \leq \rho_{\mathrm{X}}(\tau)$) for some $K > 0$.

2.5. **Some auxiliary results.** If X is a topological vector space, then we say that a subset $C \subset \mathrm{X}$ is *locally convex* if each point $x \in C$ has an open neighborhood U such that $U \cap C$ is convex.

Lemma 2.1. *Let C be a closed set of a Hausdorff topological vector space* X. *Then the following conditions are equivalent:*

(i) *C is a convex subset of* X.
(ii) *C is locally convex and connected.*

Before giving the proof we will briefly comment on the statement. We actually require a weaker form of the above lemma but this formulation appears to be the natural one. The local convexity of a *subset* must not be confused with the local convexity of a *topological vector space*, which is a completely different matter. It is easy to check that if A and B are mutually disjoint, closed, locally convex subsets of a normed (or locally convex) space X, then $A \cup B$ is locally convex subset. Clearly $A \cup B$ is not convex. Recall that the convexity of a subset can be characterized so that the subset is starlike with respect to each point of the subset. Therefore it is tempting to ask whether local convexity could be replaced by 'local starlikeness' in the above result. This is not the case as the following example shows: the 'bow tie' subspace

$$\{(x, y) \in \mathbb{R}^2 : \ |y| \leq |x| \leq 1\} \subset \mathbb{R}^2$$

is compact, connected, locally convex away from the origin, starlike with respect to the the origin, but not convex.

Proof of Lemma 2.1. The direction $(i) \implies (ii)$ is clear, because X is an open set in itself, and as a convex set of a topological vector space, it is path-connected. In order to obtain the other direction, by using the Hausdorff maximal principle, we may construct, starting from any convex subset $A \subset C$, a maximal convex subset K of C containing A. Let $\mathcal{K}$ be the set of all such maximal convex sets. Observe that the continuity of the vector operations on X yields that the closure of a convex set is again convex and thus the elements of $\mathcal{K}$ are necessarily closed sets. Our strategy is to prove that in fact $\mathcal{K} = \{C\}$.

First we check that $K_0 \cap K_1 = \emptyset$ for $K_0, K_1 \in \mathcal{K}$, $K_0 \neq K_1$. Suppose that $K_0, K_1 \in \mathcal{K}$ and $x_0 \in K_0 \cap K_1$. We show that then $K_0 = K_1$. Indeed, since K_0 and K_1 are convex sets by definitions, we need to verify that $\{tx + (1-t)y : \ t \in [0, 1]\} \subset C$ for any $x \in K_0$ and $y \in K_1$. Now, let

$$\mathrm{Cone} = \{(1-t)x_0 + t(sx + (1-s)y) : \ s, t \in [0, 1]\},$$

see Figure 1.

Then there are two possibilities. The first one is that

$$\mathrm{Cone} \cap C = \mathrm{Cone},$$

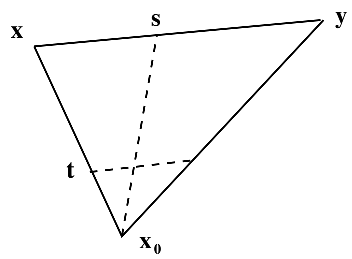

FIGURE 1. The set Cone $= \{(1-t)x_0 + t(sx + (1-s)y) : s, t \in [0,1]\}$.

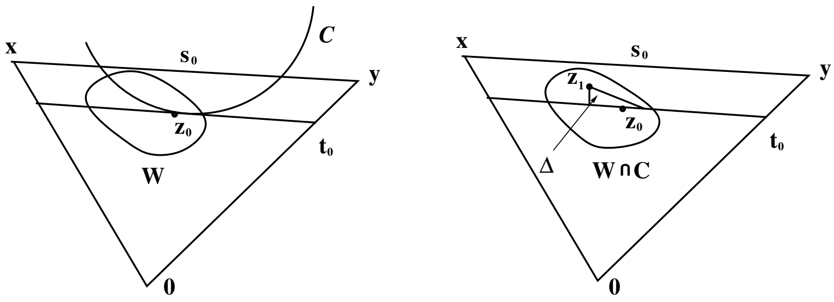

FIGURE 2. The point z_0 (left), and the set Δ (right).

in which case we have the claim. The other alternative is, by compactness, that

$$t_0 = \max\{t \in [0,1] : \{(1-t)x_0 + t(sx + (1-s)y) : s \in [0,1]\} \subset C\} < 1.$$

Without loss of generality $x_0 = 0$ and x, y are linearly independent, and our considerations are restricted to the 2-dimensional subspace $\mathrm{span}(x, y)$.

Observe that $\{tx + (1-t)x_0 : 0 \leq t \leq 1\} \subset \mathrm{Cone} \cap C$ by convexity of K_0. Let

$$s_0 = \sup\{r \in [0,1] : (1-t_0)x_0 + t_0(sx + (1-s)y) \notin \overline{\mathrm{Cone} \setminus C} \text{ for } 0 \leq s \leq r\},$$

see Figure 2. Next we apply local convexity of C at $z_0 = (1-t_0)x_0 + t_0(s_0 x + (1-s_0)y)$ to find an open neighborhood W of the point such that $W \cap C$ is convex. Pick $s_1 \leq s_0$ and $t_1 > t_0$ such that $z_1 = (1-t_1)x_0 + t_1(s_1 x + (1-s_1)y) \in W \cap C$. Then $W \cap C$ contains the convex hull Δ of

$$\{z_1\} \cup \{(1-t_0)x_0 + t_0(sx + (1-s)y) \in W \cap C : 0 \leq s \leq 1\},$$

see Figure 2.

It follows that $\mathrm{dist}(z_0, W \cap C) > 0$. This contradicts the choice of s_0. Thus $K_0 \cap K_1 = \emptyset$ for $K_0, K_1 \in \mathcal{K},\ K_0 \neq K_1$.

To verify that $\mathcal{K} = \{C\}$ we proceed as follows. Fix $K_0 \in \mathcal{K}$. Because C is connected it follows that $C \cap K_0 \cap \overline{\bigcup\{K \in \mathcal{K} : K \neq K_0\}} \neq \emptyset$, provided that $\mathcal{K}$ is not a singleton. In such a case let

$$x_0 \in C \cap K_0 \cap \overline{\bigcup \{K \in \mathcal{K} : \ K \neq K_0\}}.$$

By using the selection of x_0 and the fact that C is locally convex, we obtain a set $K_1 \in \mathcal{K},\ K_1 \neq K_0$ and an open neighborhood V of x_0 such that $C \cap (\{x_0\} \cup (V \cap K_0) \cup (V \cap K_1))$ is contained in a convex subset of C. This means, for instance, that K_0 and K_1 are connected by two line segments in C via x_0.

We claim that in fact $\mathrm{conv}(K_0 \cup (V \cap K_1)) \subset C$. This will contradict the maximality of K_0. Because K_0 and $V \cap K_1$ are convex subsets of C, we only need to show that $\{tx + (1-t)y : \ t \in [0,1]\} \subset C$ for any $x \in K_0$ and $y \in V \cap K_1$. This is seen similarly as above by studying the set Cone. Hence we obtain that $\mathcal{K} = \{C\}$, and we conclude that C is convex. □

3. Starlikeness of j-balls

Next we show that j-metric balls are starlike for radii $r \leq \log 2$.

Theorem 3.1. *Let* X *be a Banach space,* $\Omega \subsetneq \mathrm{X}$ *a domain, and let* j *be as in (2.3). Then each* j*-ball* $\mathbf{B}_j(x_0, r),\ x_0 \in \Omega$, *is starlike for radii* $r \leq \log 2$.

Proof. Let $x_0, y \in \Omega$ such that $j(x_0, y) \leq \log 2$. This is to say that

$$\frac{\|x_0 - y\|}{d(x_0) \wedge d(x)} \leq 1.$$

By using simple calculations and the triangle inequality we get

$$\begin{aligned} j(x_0, ty + (1-t)x_0) &= \log\left(1 + \frac{\|x_0 - (ty + (1-t)x_0)\|}{d(x_0) \wedge d(ty + (1-t)x_0)}\right) \\ &\leq \log\left(1 + \frac{(1-t)\|x_0 - y\|}{d(x_0) \wedge (d(y) - t\|x_0 - y\|)}\right) \leq \log 2, \end{aligned}$$

where we applied the fact $d(x_0), d(y) \geq \|x_0 - y\|$ in the last inequality. □

Proposition 3.2. *Let* X *be a Banach space and* $\Omega \subset \mathrm{X}$ *a domain with* $\partial\Omega \neq \emptyset$. *Then* $\mathbf{B}_{j_\Omega}(x, r) = \bigcap_{z \in \mathrm{X}\setminus\Omega} \mathbf{B}_{j_{\mathrm{X}\setminus\{z\}}}(x, r)$. *Moreover, if* X *is reflexive and* Ω *is weakly open, then*

$$\mathbf{U}_{j_\Omega}(x, r) = \bigcap_{z \in \mathrm{X}\setminus\Omega} \mathbf{U}_{j_{\mathrm{X}\setminus\{z\}}}(x, r).$$

Proof. Denote by C the norm closed set $\mathrm{X} \setminus \Omega$. First note that $\mathrm{X} \setminus C \subset \mathrm{X} \setminus \{z\}$ and that $j_{\mathrm{X}\setminus\{z\}} \leq j_\Omega$ holds on Ω for each $z \in C$. Thus

$\mathbf{B}_{j_\Omega}(x,r) \subset \bigcap_{z\in C} \mathbf{B}_{j_{\mathrm{X}\setminus\{z\}}}(x,r)$ and $\mathbf{U}_{j_\Omega}(x,r) \subset \bigcap_{z\in C} \mathbf{U}_{j_{\mathrm{X}\setminus\{z\}}}(x,r)$. Pick $y \in \Omega$ such that

$$j(x,y) = \log\left(1 + \frac{\|x-y\|}{d(x)\wedge d(y)}\right) > r.$$

Then there is $z \in C$ such that

$$\log\left(1 + \frac{\|x-y\|}{\|x-z\|\wedge\|y-z\|}\right) > r.$$

This means that $y \notin \bigcap_{z\in C} \mathbf{B}_{j_{\mathrm{X}\setminus\{z\}}}(x,r)$ and so we have the first part of the statement.

Now, assume that X is reflexive and Ω is weakly open. Pick $y \in \Omega$ with $j(x,y) = r_0 \geq r$. Let $v \in \{x,y\}$ and $s_0 \in \mathbb{R}$ be such that

$$r_0 = \log\left(1 + \frac{\|x-y\|}{d(v)}\right) = \log\left(1 + \frac{\|x-y\|}{s_0}\right).$$

Note that C is weakly closed and thus by James' well-known characterization of reflexivity of Banach spaces (see e.g. [2]) we get that $\mathbf{B}_{\|\cdot\|}(v, s_0+1)\cap C$ is weakly compact. Thus $\bigcap_{\epsilon>0} \mathbf{B}_{\|\cdot\|}(v, s_0+\epsilon)\cap C \neq \emptyset$, so let us select a point z from this set. Note that $\|v-z\| = s_0$, since $d(v) = s_0$. This means that

$$j_{\mathrm{X}\setminus\{z\}}(x,y) \geq \log\left(1 + \frac{\|x-y\|}{\|v-z\|}\right) = r_0 \geq r.$$

Consequently, $\mathbf{U}_{j_\Omega}(x,r) \subset \bigcap_{z\in C} \mathbf{U}_{j_{\mathrm{X}\setminus\{z\}}}(x,r)$. □

Remark 3.3. The quasihyperbolic metric on $\mathrm{X}\setminus\{0\}$ is conformal in the following sense: for each $C > 1$ there is $r > 0$ such that

$$C^{-1}k(x,y) \leq \frac{\|x-y\|}{\|x\|} \leq Ck(x,y)$$

for $k(x,y) < r$. The same is true for the distance ratio metric. Note that we did not assume anything about the geometry of X. The proof follows the arguments in [18, p. 35], and is left to the reader.

Remark 3.4. Klén's main results in [5] and [6] involving $\mathbb{R}^n$ can be adapted to general (finite-dimensional, separable, non-separable, real or complex) Hilbert spaces H. This is due to the fact that the core of the arguments is, roughly speaking, based on calculations in $\mathbb{R}^2$ and then these observations extend to $\mathbb{R}^n$ by elegant reasoning. Essentially the same extension carries further to Hilbert spaces.

4. Convexity of quasihyperbolic and j-balls on convex domains

In this section, we study convexity of quasihyperbolic and j-metric balls. We present a generalization of a result of Martio and Väisälä [9, 2.13].

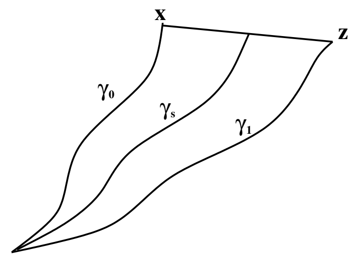

FIGURE 3. The average path γ_s.

Theorem 4.1. *Let* X *be a Banach space and* $\Omega \subsetneq$ X *a convex domain. Then all quasihyperbolic balls and* j*-balls on* Ω *are convex. Moreover, if* Ω *is uniformly convex, or if* X *is strictly convex and has the RNP, then these balls are strictly convex.*

Fact 4.2. *Let* $a, b, c, d > 0$ *be constants such that* $a/c = b/d$. *Then*

$$\frac{ta + (1-t)b}{tc + (1-t)d} = \frac{a}{c} \quad \text{for } t \in [0,1].$$

Proof. This fact can be verified by differentiating with respect to t. □

Proof of Theorem 4.1. We will prove the case with the quasihyperbolic metric, which is more complicated. Fix $x \in \Omega$ and $r > 0$. Let $y, z \in D(x,r)$. Our aim is to verify that $sy + (1-s)z \in D(x,r)$ for $s \in [0,1]$. Thus, we may assume that $k(x,y) = k(x,z) = r$ in the first place. By using suitable translations we may assume that $x = 0$ as well. It suffices to show that

$$k(x, sy + (1-s)z) \leq r, \quad \text{for } s \in [0,1]. \tag{4.1}$$

We use the following short-hand notation

$$\ell_k(\gamma, t_1, t_2) = \int_{t_1}^{t_2} \frac{\|d\gamma(t)\|}{d(\gamma(t))},$$

where $\gamma\colon [0,1] \to$ X is a rectifiable path and $0 \leq t_1 \leq t_2 \leq 1$. We will also write $\ell_k(\gamma)$ instead of $\ell_k(\gamma, 0, 1)$.

Let $\epsilon > 0$ and let $\gamma_0, \gamma_1\colon [0,1] \to$ X be rectifiable paths such that $\gamma_0(0) = \gamma_1(0) = 0$, $\gamma_0(1) = z$, $\gamma_1(1) = y$, $\ell_k(\gamma_0) \leq r + \epsilon$ and $\ell_k(\gamma_1) \leq r + \epsilon$. We may assume by symmetry that $\ell_k(\gamma_0) \leq \ell_k(\gamma_1)$. Moreover, modifying by re-parameterizing γ_0 suitably, we may assume that $\ell_k(\gamma_0) = \ell_k(\gamma_1)$ and $\ell_k(\gamma_0, 0, t) = \ell_k(\gamma_1, 0, t) = t\ell_k(\gamma_0)$ for $t \in [0,1]$. Thus we have that

$$\frac{\|D\gamma_0(t)\|}{d(\gamma_0(t))} = \frac{\|D\gamma_1(t)\|}{d(\gamma_1(t))} = \ell_k(\gamma_0) = \ell_k(\gamma_1) \quad \text{for a.e. } t \in [0,1]. \tag{4.2}$$

Observe that the above numerators need not be continuous, so that these terms do not coincide, at least a priori, for every t.

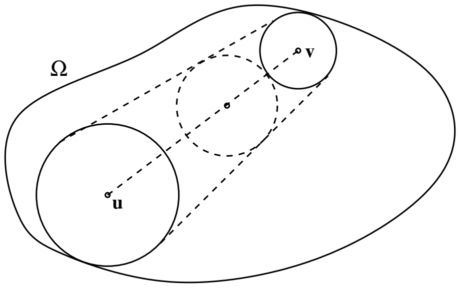

FIGURE 4. The ball $\mathbf{U}_{\|\cdot\|}(sv + (1-s)u, sd(v) + (1-s)d(u))$.

Define an average path γ_s (see Figure 3) for $s \in [0,1]$ by $\gamma_s(\cdot) = s\gamma_1(\cdot) + (1-s)\gamma_0(\cdot)$. Clearly $\gamma(0)_s = 0$ and $\gamma_s(1) = sy + (1-s)z$ for $s \in [0,1]$. We claim that

$$\ell_k(\gamma_s(\cdot)) \leq s\ell_k(\gamma_1(\cdot)) + (1-s)\ell_k(\gamma_0(\cdot)) = \ell_k(\gamma_0) = \ell_k(\gamma_1). \tag{4.3}$$

Because ϵ was arbitrary this estimate yields (4.1), which provides the required result.

To obtain the estimate (4.3), observe that

$$\|D\gamma_s(\cdot)\| \leq s\|D\gamma_1(\cdot)\| + (1-s)\|D\gamma_0(\cdot)\|, \quad 0 \leq s \leq 1 \tag{4.4}$$

holds point-wise by the triangle inequality as we recall the definition of the norm length ℓ. Given $v, u \in \Omega$ it holds that

$$\mathbf{U}_{\|\cdot\|}(v, d(v)) \cup \mathbf{U}_{\|\cdot\|}(u, d(u)) \subset \Omega$$

and by the convexity of Ω it holds that

$$\{sa + (1-s)b : \ a \in \mathbf{U}_{\|\cdot\|}(v, d(v)), \ b \in \mathbf{U}_{\|\cdot\|}(u, d(u)), \ s \in [0,1]\} \subset \Omega.$$

Moreover, the above set contains $\mathbf{U}_{\|\cdot\|}(sv+(1-s)u, sd(v)+(1-s)d(u))$, see Figure 4. See also [15, Lemma 3.5].

This means that

$$d(su + (1-s)v) \geq sd(u) + (1-s)d(v). \tag{4.5}$$

Now, by combining (4.4), (4.5), (4.2) and Fact 4.2 we obtain

$$\ell_k(\gamma_s) = \int_0^1 \frac{\|d\gamma_s(t)\|}{d(\gamma_s(t))} \leq \int_0^1 \frac{s\|d\gamma_1(t)\| + (1-s)\|d\gamma_0(t)\|}{d(\gamma_s(t))}$$

$$\leq \int_0^1 \frac{s\|d\gamma_1(t)\| + (1-s)\|d\gamma_0(t)\|}{sd(\gamma_1(t)) + (1-s)d(\gamma_0(t))} = \int_0^1 \ell_k(\gamma_0)\, dt = \ell_k(\gamma_0).$$

This completes the proof for the first part of the statement.

In the latter part, suppose that $\gamma_0 \neq \gamma_1$. Then $\gamma_s(t),\ 0 < s < 1$,

- Satisfies (4.5) strictly for a set of values of t having positive measure if Ω is uniformly convex.
- Satisfies (4.4) strictly for a set of values of t having positive measure if X is strictly convex and has the RNP.

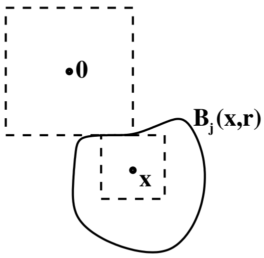

FIGURE 5. There is no critical radius $R > 0$ such that the ball $\mathbf{B}_j(x, r)$ is convex for all $x \in \Omega = \ell^\infty(2) \setminus \{0\}$ and $0 < r < R$.

The strict convexity of the QH-balls follows. □

5. Convexity of balls in a punctured Banach space

In this section we study convexity of the balls with respect to the quasihyperbolic and the distance ratio metrics.

Fact 5.1. *Let $x, y > 0$ and $a, b, c, d \in \mathbb{R}$ such that $a + b \geq c + d$ and $y + c, y + d > 0$. Then*

$$\max\left\{\frac{x+a}{y+c}, \frac{x+b}{y+d}\right\} \geq \frac{x}{y}.$$

Proof. This fact follows easily, as one may assume without loss of generality that

$$\frac{x+a}{y+c} \leq \frac{x+b}{y+d}.$$

□

Theorem 5.2. *Let* X *be a Banach space, which is uniformly smooth and uniformly convex, both of power type* 2. *Consider* $\Omega = \mathrm{X} \setminus \{0\}$ *endowed with the j-metric. Then there exists a constant $R > 0$ such that all j-balls of radius $r \leq R$ are convex.*

Proof. Without loss of generality it suffices to consider balls $\mathbf{B}_j(x_0, r_0)$ with $x_0 \in \mathrm{X}$ such that $\|x_0\| = 1 + r$, where we use the shorthand notation $r = e^{r_0} - 1$. Then

$$\mathbf{B}_j(x_0, r_0) = \left\{x \in \mathrm{X} : \frac{\|x - x_0\|}{\|x\|} \leq r\right\} \cap \left\{x \in \mathrm{X} : (1+r)^{-1}\|x - x_0\| \leq r\right\},$$

where the right-most set of the intersection is clearly convex. It follows that we need to verify that the sets

(5.1)

$$A = \left\{x \in \mathrm{X} : \frac{\|x - x_0\|}{\|x\|} < r,\ 1 \leq \|x\| \leq (1 + r)^2\right\}, \quad 0 < r < R,$$

are convex as well for a suitable choice of $R > 0$. The selection of R is discussed at the end of the proof in more detail.

Since X is uniformly convex and smooth of power type 2, it is easy to check that there exists $M > 1$ such that

$$\liminf_{h\to 0^+} \frac{(th)^{-1}\inf\{\|p+z\|+\|p-z\|-2:\ \|p\|=1,\ \|z\|=th\}}{h^{-1}\sup\{\|v+w\|+\|v-w\|-2:\ \|v\|=1,\ \|w\|=h\}} \geq 1, \tag{5.2}$$

for $t \geq M$. Fix such $M > 1$.

Note that $\mathbf{B}_j(x_0, r_0),\ 0 < r_0 < \log 2$, is starlike by Theorem 3.1 and hence connected. According to Lemma 2.1 it is only required to verify that A is locally convex in small neighborhoods at the boundary. By using a compactness argument for 2-dimensional sections, similar as employed in the proof of Lemma 2.1, it follows that if A is not locally convex at the boundary, then the following holds: there exists $x \in A$ such that

$$\frac{\|x-x_0\|}{\|x\|} = r$$

we have for some $y \in \mathbf{S}_{\mathrm{X}}$ the inequality

$$\inf_{0<h<H} \max\left\{\frac{\|x+hy-x_0\|}{\|x+hy\|}, \frac{\|x-hy-x_0\|}{\|x-hy\|}\right\} - r < 0 \tag{5.3}$$

for *all* $H > 0$. Next we aim to exclude this possibility.

Indeed, write $t = \|x-x_0\|^{-1},\ s = \|x\|^{-1}$ and use $p = t(x-x_0),\ z = thy,\ v = sx,\ w = shy$ in (5.2) to obtain that

$$\liminf_{h\to 0^+} \frac{\|t(x-x_0)+thy\|+\|t(x-x_0)-thy\|-2}{\|sx+shy\|+\|sx-shy\|-2} \geq \frac{t}{s}$$

for $t/s \geq M$ and hence

$$\liminf_{h\to 0^+} \frac{\|(x-x_0)+hy\|+\|(x-x_0)-hy\|-2\|x-x_0\|}{\|x+hy\|+\|x-hy\|-2\|x\|} \geq 1. \tag{5.4}$$

By Fact 5.1 and (5.4) we have

$$\inf_{0<h<H} \max\left\{\frac{\|x-x_0+hy\|}{\|x+hy\|}, \frac{\|x-x_0-hy\|}{\|x-hy\|}\right\} \geq \frac{\|x-x_0\|}{\|x\|}$$

for sufficiently small $H > 0$. This contradicts (5.3). The constant $R > 0$ is obtained as follows. Because it was required that

$$\frac{t}{s} = \frac{\|x\|}{\|x-x_0\|} \geq M,$$

taking into account (5.1), it suffices to put $R = M^{-1}$. □

Corollary 5.3. *Let* X *be a Banach space, which is uniformly smooth and uniformly convex, both of power type* 2*. Consider a domain* $\Omega \subsetneq \mathrm{X}$ *endowed with the* j*-metric. Then there exists a constant* $R > 0$ *such that all* j*-balls of radius* $r \leq R$ *are convex.*

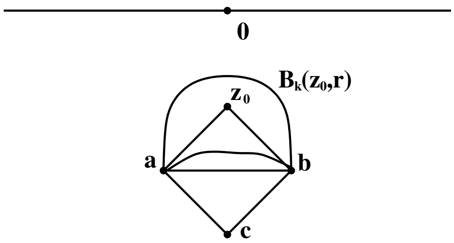

FIGURE 6. The path γ_0 consists of line segments $[a, c]$ and $[c, b]$.

Proof. We apply the previous result together with Proposition 3.2 and the standard method of [6] applied in passing from punctured spaces to general domains. □

In [9, 2.14] Martio and Väisälä asked whether the quasihyperbolic balls of convex domains of uniformly convex Banach spaces are quasihyperbolically convex. More precisely, given two points a and b of the quasihyperbolic ball $B \subset \Omega$, does there exist a geodesic γ joining a and b, which is contained in the ball B. Here the domain Ω was assumed to be convex and the length of the geodesic is measured with respect to the quasihyperbolic metric. It turns out the the answer is negative, as the following counterexample shows.

Example 5.4. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : \ y < 0\}$ and we will first consider Ω as a subset of $\ell^\infty(2) = (\mathbb{R}^2, \|\cdot\|_\infty)$. Let $x = (0, -1)$,

$$r = \ln(2) = \int_1^2 t^{-1}\, dt.$$

We will study the ball $\mathbf{B}_k(x, r)$. Put $a = (-1, -2)$, $b = (1, -2)$ and observe that $\{ta + (1-t)b : \ t \in [0, 1]\}$ is included in $\partial\mathbf{B}_k(x, r)$. An intuition, which helps in computing the quasihyperbolic lengths of paths, is that one can move to the directions $(-1, -1)$, $(0, -1)$ and $(1, -1)$ at the same cost because of the choice of the norm. Note that $z_2 \geq -2$ for any $(z_1, z_2) \in \mathbf{B}_k(x, r)$.

Now, an easy computation shows that any path $\gamma \subset \mathbf{B}_k(x, r)$, which joins a and b must have quasihyperbolic length at least

$$\int_{-1}^1 \frac{1}{2}\, dt = 1.$$

However, the broken line γ_0 connecting a, b through the point $c = (0, -3)$ has length

$$2\int_0^1 \frac{1}{3-t}\, dt = \ln\left(\frac{9}{4}\right) < 1,$$

see Figure 6. The existence of geodesics is clear in this choice of space. Thus $\mathbf{B}_k(x, r)$ is not quasiconvex.

This example does not change considerably if one considers the domain $\Omega = (-6, 6) \times (0, 6)$ instead. Observe that the space $\ell^\infty(2)$ is certainly not uniformly convex, see Figure 5. However, because the quasihyperbolic metric depends continuously on the selection of the norm, we could apply the space $\ell^p(2)$ for large $p < \infty$ in place of $\ell^\infty(2)$ to produce similar examples, in which case we are dealing with uniformly convex spaces.

5.1. **Convexity of quasihyperbolic balls in a punctured Banach space.** Next we generalize the work of Klén [5], mutatis mutandis, to the Banach space setting.

Lemma 5.5. *Let $f \in L^2$ such that $f \neq 0$ a.e. and let $F(t) = \int_0^t f(s)\ ds$, $0 \leq t \leq 1$. Then*

$$\frac{\int_0^t F(s)^2\ ds}{\int_0^t f(s)^2\ ds} \leq t^2 \quad \text{for } 0 \leq t \leq 1.$$

Proof. We will apply the well-known fact that the expectation operator on $L^2([0,t])$ is contractive, which is easiest to see by writing it like $1 \otimes 1$. Then we have

$$\frac{\int_0^t F(s)^2\ ds}{\int_0^t f(s)^2\ ds} \leq \frac{tF(t)^2}{\int_0^t (F(t)/t)^2\ ds} = \frac{tF(t)^2}{F(t)^2/t} = t^2.$$

□

In the above lemma it is essential that the exponents appearing in the numerator and the denominator are the same. This can be seen by multiplying f with suitable positive constants, as F depends linearly on f.

Lemma 5.6. *Let X be a Banach space, which is uniformly convex and uniformly smooth, both moduli being of power type 2. We consider the quasihyperbolic metric k on $\mathrm{X} \setminus \{0\}$. Then there exists $R > 0$ as follows. Assume that $\gamma_1, \gamma_2 \colon [0, t_2] \to \mathrm{X} \setminus \{0\}$ are rectifiable paths satisfying the following conditions:*

(i) γ_1, γ_2 *and* $\frac{\gamma_1+\gamma_2}{2}$ *are contained in* $\mathbf{B}_{\|\cdot\|}(0,2) \setminus \mathbf{B}_{\|\cdot\|}(0,1)$,
(ii) $\gamma_1(0) = \gamma_2(0)$,
(iii) $\ell_k(\gamma_1) \vee \ell_k(\gamma_2) \leq R$
(iv) $\ell_{\|\cdot\|}(\gamma_1) = t_1 \leq t_2 = \ell_{\|\cdot\|}(\gamma_2)$
(v) *The paths are parameterized with respect to* $\ell_{\|\cdot\|}$, *except that* $\gamma_1(t) = \gamma_1(t_1)$ *for* $t \in [t_1, t_2]$.

Then the following estimate holds:

$$\frac{\ell_k(\gamma_1) + \ell_k(\gamma_2)}{2} \geq \ell_k\left(\frac{\gamma_1+\gamma_2}{2}\right) + \int_0^{t_1} \frac{\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|}\ ds.$$

Proof. We note that the assumption about the parameterization yields that

$$\|D\gamma_1(t)\| = \|D\gamma_2(t)\| = 1 \quad \text{for } t \in [0, t_1].$$

Recall that we denote the Gâteaux derivative of a path γ by $D\gamma$. Since X has the RNP, being a reflexive space, it follows that each reasonably parameterized path of finite QH-length is differentiable almost everywhere and can be recovered from its derivative by Bochner integration.

By using assumption (i) we observe that

$$\begin{aligned}\ell_k\left(\frac{\gamma_1+\gamma_2}{2}, t_1, t_2\right) &= \int_{t_1}^{t_2} \frac{\|D(\frac{\gamma_1+\gamma_2}{2})\|}{\|\frac{\gamma_1+\gamma_2}{2}\|}\, ds = \int_{t_1}^{t_2} \frac{\|D\gamma_2\|}{2\|\frac{\gamma_1(t_1)+\gamma_2}{2}\|}\, ds \\ &\leq \int_{t_1}^{t_2} \frac{\|D\gamma_2\|}{2}\, ds \leq \int_{t_1}^{t_2} \frac{\|D\gamma_2\|}{\|\gamma_2\|}\, ds = \ell_k(\gamma_2, t_1, t_2).\end{aligned}$$

Thus our task reduces to verifying that

$$\frac{\ell_k(\gamma_1, 0, t_1) + \ell_k(\gamma_2, 0, t_1)}{2} \geq \ell_k\left(\frac{\gamma_1+\gamma_2}{2}, 0, t_1\right) + \int_0^{t_1} \frac{\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|}\, ds.$$

Without loss of generality we may assume, possibly by re-defining the paths, that $\|D(\gamma_1 - \gamma_2)(t)\|$ is not zero in any open neighborhood of 0.

Let us evaluate by using the convexity of the mapping $t \mapsto t^{-1}$ and the moduli of smoothness and convexity in the following manner:

$$\begin{aligned}&\frac{1}{2}\left(\frac{\|D\gamma_1\|}{\|\gamma_1\|} + \frac{\|D\gamma_2\|}{\|\gamma_2\|}\right) = \frac{1}{2}\left(\frac{1}{\|\gamma_1\|} + \frac{1}{\|\gamma_2\|}\right) \\ \geq\ & \frac{2}{\|\gamma_1\| + \|\gamma_2\|} \\ \geq\ & \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1\| + \|\gamma_2\|} + \frac{2\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|} \\ \geq\ & \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|(1+2\rho_{\mathrm{X}}(\|\gamma_1-\gamma_2\|/2\|\gamma_1+\gamma_2\|))} + \frac{2\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|}.\end{aligned}$$

We aim to verify that there exists $R > 0$ such that

$$\begin{aligned}&\int_0^t \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|\big(1+2\rho_{\mathrm{X}}\big(\|\gamma_1-\gamma_2\|/2\|\gamma_1+\gamma_2\|\big)\big)} + \frac{2\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|}\, ds \\ \geq\ & \int_0^t \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|} + \frac{\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\| + \|\gamma_2\|}\, ds\end{aligned}$$

for all $0 \leq t \leq R$. Recall that $1 \leq \|\gamma_1 + \gamma_2\| \leq 4$ by the assumptions. Let us analyze the terms of the above inequality:

$$\begin{aligned}
&\int_0^t \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|} ds - \int_0^t \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|\big(1+2\rho_{\mathrm{X}}\big(\|\gamma_1-\gamma_2\|/2\|\gamma_1+\gamma_2\|\big)\big)} ds \\
= &\int_0^t \frac{\|D(\gamma_1+\gamma_2)\|}{\|\gamma_1+\gamma_2\|}\left(1-\frac{1}{(1+2\rho_{\mathrm{X}}(\|\gamma_1-\gamma_2\|/2\|\gamma_1+\gamma_2\|))}\right) ds \\
\leq &\int_0^t \left(1-\frac{1}{(1+2\rho_{\mathrm{X}}(\|\gamma_1-\gamma_2\|/8))}\right) ds \leq \int_0^t 2\rho_{\mathrm{X}}(\|\gamma_1-\gamma_2\|/8)\ ds
\end{aligned}$$

and

$$\int_0^t \delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)/8\ ds \leq \int_0^t \frac{\delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)}{\|\gamma_1\|+\|\gamma_2\|}\ ds.$$

To justify the existence of the claimed constant $R > 0$ it suffices to check that

$$\frac{\int_0^t 2\rho_{\mathrm{X}}(\|\gamma_1-\gamma_2\|/8)\ ds}{\int_0^t \delta_{\mathrm{X}}(\|D(\gamma_1-\gamma_2)\|)/8\ ds} \longrightarrow 0 \tag{5.5}$$

uniformly, regardless of the selection of paths, as $t \to 0$.

Define $f(s) = \|D(\gamma_1 - \gamma_2)(s)\|$ for a.e. $s \in [0, r]$ and put

$$F(t) = \int_0^t f(s)\ ds \geq \|\gamma_1(t) - \gamma_2(t)\|.$$

Recall that $\rho_{\mathrm{X}}(\tau) \leq K\tau^2$ and $\delta_{\mathrm{X}}(\epsilon) \geq M\epsilon^2$. Then the above ratio in (5.5) can be evaluated from above by

$$\frac{\int_0^t 2\rho(F(s)/8)\ ds}{\int_0^t \delta_{\mathrm{X}}(f(s))/8\ ds} \leq (2)^{-2}M^{-1}K\frac{\int_0^t F(s)^2\ ds}{\int_0^t f(s)^2\ ds} \leq (2)^{-2}M^{-1}Kt^2. \tag{5.6}$$

Above we applied Lemma 5.5 and we note that the right-hand side tends to 0 as $t \to 0$, independently of the choice of f. Thus we have the claim. □

Theorem 5.7. *Let* X *be a Banach space, which is uniformly convex and uniformly smooth, both moduli being of power type* 2. *We consider the quasihyperbolic metric* k *on* $\mathrm{X} \setminus \{0\}$. *Then there exists* $R > 0$ *as follows:*

(i) *Each quasihyperbolic ball* $\mathbf{B}_k(x, r)$, $r \leq R$, *is strictly convex.*
(ii) *For each* $y \in \mathbf{B}_k(x, r)$, $r \leq R$, *there exists a unique geodesic in* $\mathbf{B}_k(x, r)$ *joining* x *to* y.
(iii) *Suppose that* $v_n, w_n, y_n, z_n \in \mathbf{B}_k(x, r)$, $r \leq R$, $n \in \mathbb{N}$, *and* $\lambda_n, \gamma_n \subset \mathbf{B}_k(x, r)$ *are paths of finite quasihyperbolic length and parameterized so that they have a constant norm length growth.*

Assume further that $\gamma_n(0) = v_n$, $\lambda_n(0) = w_n$, $\gamma_n(1) = y_n$, $\lambda_n(1) = z_n$ *and*

$$\lim_{n\to\infty} k(v_n, w_n) = \lim_{n\to\infty} k(y_n, z_n) = 0,$$

and

$$\lim_{n\to\infty} \ell_k(\gamma_n) - k(v_n, y_n) = \lim_{n\to\infty} \ell_k(\lambda_n) - k(w_n, z_n) = 0.$$

Then

$$\lim_{n\to\infty} \ell_k(\gamma_n - \lambda_n) = 0.$$

Proof. After normalizing with suitable quasihyperbolic isometries, bearing Remark 3.3 in mind, the constant R is obtained from Lemma 5.6. We will begin by proving claim (iii). Without loss of generality we may assume, by extending and re-parameterizing the paths γ_n, that $\gamma_n(0) = \lambda_n(0)$, $\gamma_n(1) = \lambda_n(1)$ and $\ell_k(\gamma_n) = \ell_k(\lambda_n)$ for each n. It is easy to see that the difference between the original and modified version of γ_n tend to 0 in terms of the quasihyperbolic length as $n \to \infty$.

Since $\lim_{n\to\infty} \ell_k(\gamma_n) - \ell_k(\lambda_n) = 0$, we obtain by Lemma 5.6 applied to $(\gamma_n + \lambda_n)/2$ that

$$\int_0^{t_n} \frac{\delta_{\mathrm{X}}(\|D(\gamma_n - \lambda_n)\|)}{\|\gamma_n\| + \|\lambda_n\|}\, ds \to 0 \quad \text{as } n \to \infty,$$

where $t_n = \ell_{\|\cdot\|}(\gamma_n) \wedge \ell_{\|\cdot\|}(\lambda_n)$. By using the fact that the norm of the elements of $\mathbf{B}_k(x, r)$ is bounded from above and the modulus of convexity is of power type 2, we get that

$$\int_0^{t_n} \|D(\gamma_n - \lambda_n)\|\, ds \to 0 \quad \text{as } n \to \infty.$$

On the other hand, since the norm of the elements of $\mathbf{B}_k(x, r)$ is bounded from below by a strictly positive constant, we obtain that $\ell_k(\gamma_n - \lambda_n)$ tends to 0 as $n \to \infty$.

To verify claim (ii), fix $y \in \mathbf{B}_k(x, r)$. Let γ_n be a sequence of rectifiable paths $I \to \mathrm{X} \setminus \{0\}$ parametrized with respect to $\ell_{\|\cdot\|}$ such that $\gamma_n(0) = x$, $\gamma_n(t) = y$ for $t \in [\ell_{\|\cdot\|}(\gamma_n), 1]$ and $\ell_k(\gamma_n) \to k(x, y)$ as $n \to \infty$. A similar reasoning as above yields that

$$\sup_k \int_0^{t_{n,k}} \|D(\gamma_n - \gamma_{n+k})\|\, ds \to 0 \quad \text{as } n \to \infty.$$

According to the RNP we may consider the weak derivative $D\gamma$ of a path γ as an element of the Bochner space $L^1([0,1], \mathrm{X})$, where the $\ell_{\|\cdot\|}$ norm of $\gamma(\cdot) - x$ coincides with the Bochner norm of $D\gamma \in L^1([0,1], \mathrm{X})$. Thus $(D\gamma_n) \subset L^1([0,1], \mathrm{X})$ is a Cauchy sequence, and since the Bochner space is complete, we may let $D\gamma \in L^1([0,1], \mathrm{X})$ be the (unique) point of convergence. Then we obtain the required geodesic γ by defining it as a Bochner integral as follows:

$$\gamma(t) = x + \int_0^t D\gamma \; ds.$$

It is straight-forward to check that γ is a geodesic. Moreover, since the Bochner space element $D\gamma$ is unique, in the sense that it is independent of the selection of the sequence $D\gamma_n$ (as long as $\ell_k(\gamma_n) \to k(x,y)$ as $n \to \infty$), we conclude that γ is unique as well.

Let us verify claim (i) that $\mathbf{B}_k(x,r)$ is strictly convex. Fix two points $y, z \in \mathrm{X}$, $y \neq z$, such that $k(x,y) = k(x,z) = r$. There exist quasihyperbolic geodesics γ and λ, joining x, y and x, z, respectively. By using Lemma 5.6 we obtain that

$$\ell_k\left(\frac{\gamma + \lambda}{2}\right) < r,$$

and clearly the average path $(\gamma + \lambda)/2$ joins x with $(y+z)/2$. This completes the proof. □

Any Hilbert space has the best possible power types of uniform convexity and uniform smoothness, namely $p = 2$, and in fact the optimal modulus functions. It is known that any Banach space has the uniform convexity power type at least 2 and the uniform smoothness power type at most 2. Our method in the proof of Lemma 5.6 requires comparing the power types and this is why we assumed that the power types of the moduli should coincide, i.e. $p = 2$ for both accounts. It is perhaps worthwhile to pay close attention to how Lemma 5.5 is applied at the end of the proof. We note that any Banach space with the coinciding power types of the moduli must be linearly homeomorphic to a Hilbert space.

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