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A.A. El-Sabbagh F.A. Abd El Salam K. El Nagaar

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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A.A. El-Sabbagh F.A. Abd El Salam K. El Nagaar

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *For regular canonical system of first order differential equations, we associate symmetric linear relations. Also, we define the minimal and the maximal relations for this case and construct the generalized resolvents related to the selfadjoint extensions for these symmetric relations defined in Hilbert space H or larger than the given space H. One can construct the eigenfunction expansions for the systems we are interested in and the so-called Weyl-coefficients which are the main idea to construct the spectrum of the symmetric relations. We may illustrate this case by giving some examples.*

AMS subject classifications: 47A20

Keywords: Hilbert Spaces, Symmetric Relations, Minimal and Maximal relations, Weyl-Coefficients, and Canonical Systems of Differential Equations.

Correspondence

Department of Mathematics
Faculty of Engineering
Benha University, Shoubra, 108 Shoubra Street
Cairo, Egypt.

alysab1@hotmail.com.

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FI-02015 TKK, Finland
email:math@tkk.fi http://math.tkk.fi/

1 Introduction

We shall consider canonical systems of first order differential expressions regular on the compact interval $[a, b]$. For a given symmetric linear relation S in a Hilbert space H, the selfadjoint extensions of S can be characterized as restrictions of the adjoint S^* of S, when S is the minimal relation associated with a formally symmetric ordinary differential expression in L^2-function space, then the restrictions involve linear combinations of the boundary values of the elements in the domain $D(S^*)$ of S^*. When the selfadjoint extensions are canonical within the space H, the coefficients of these combinations can be taken to be constants. In the case of selfadjoint extensions in inner product spaces larger than the given space H, they depend analytically on a parameter, see [9], [11], and [18]. We shall prove that every generalized resolvent $R(\ell)$ of S can be expressed in terms of a fixed generalized resolvent $G(\ell)$ of S and the Weyl coefficients $\Psi(\ell)$ of $R(\ell)$ relative to $G(\ell)$ as

$$R(\ell)f = G(\ell)f + s(\ell)\Psi(\ell)[f, S(\ell)], f \in H \tag{1}$$

where $s(\ell)$ is a holomorphic basis for the null space $v(S^* - \ell)$ see [15], [16], and the spectrum of S can be constructed. Finally, we give examples; some in the classical boundary value problem and the others are the general boundary value problem.

2 Eigenfunction Expansion

Let A be a selfadjoint extension of S in a Krein space K, with nonempty resolvent set $\rho(A) = \{\ell \in C \mid (A - \ell)^{-1} \in [K]\}$. By definition $H \subset K$, and the definite and indefinite inner products on H and K, respectively, coincide on H. In the sequel we only consider selfadjoint extensions, whose resolvent sets are nonempty. By PH we denote the orthogonal projection of K onto H. We say that A or K is minimal if $K = c.l.s\{\{(A - \ell)^{-1}K \mid \ell \in \rho(A)\} \cup H\}$ where $c.l.s.$ stands for closed linear span. The compressed resolvent $R(\ell), \ell \in \rho(A)$ associated with the extension A is defined by

$$R(\ell) = P_H(A - \ell)^{-1} \mid_H \tag{2}$$

see [1], [3], [6] and [24].

Theorem 2.1 *The compressed resolvent $R(\ell)$, defined in 2 satisfies:*
(a) $R(\ell)$ is a holomorphic mapping with values in H and with domain of holomorphy D_R, which is symmetric with respect to the real axis, $D_R = D_R^*$
(b) $R(\ell)^* = R(\bar{\ell})$
(c) $\{R(\ell)f, \ell R(\ell)f + f\} \in S^*$ for all $f \in H$

Proof . Let $F(\ell) \in \mathcal{L}(N)$ have representation (2), then for $\lambda \in \mathbb{C}\backslash\mathbb{R}$

$$(R_\lambda F)(\ell) = \int_R \frac{d\Sigma(t)f(t)}{(t - \ell)(t - \lambda)}, \tag{3}$$

implies that $R_\lambda F \in \mathcal{L}(N)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. By (2) & (3)

$$\begin{aligned}
\|F_\lambda F\|^2 &= \|f(t)/(t-\lambda)\|_\Sigma^2 = \int_R \frac{f(t)d\Sigma(t)f(t)}{(t-\ell)} d\Sigma(t)\left(\frac{f(t)}{(t-\lambda)}\right) \\
&\leq |\mathrm{Im}\lambda|^{-2} \int_R f(t)^* d\Sigma(t) f(t) \\
&= |\mathrm{Im}\lambda|^{-2} \parallel f \parallel_\Sigma^2 \leq |\mathrm{Im}\lambda|^{-2} \|F\|^2,
\end{aligned}$$

for $\lambda \in \mathbb{C}\backslash\mathbb{R}$, which shows that R_λ is a bounded operator in $\mathcal{L}(N)$. This proves (a) and (b) Combining (a) and (b), we obtain the identity $R_\lambda - R_\mu^* = (\lambda - \overline{\mu})R_\mu^* R_\lambda, \quad \lambda, \mu \in \mathbb{C}\backslash\mathbb{R}$

Property (c) can be written as $I \subset (S^* - \ell)R(\ell)$ (with equality if S is densely defined), where I is the identity on H. Because of (b), it is equivalent to $R(\ell)(S-\ell) \subset I$. It is not difficult to verify that for each $\ell \in D_R$

$$S^* = \{\{R(\ell)f, \ell R(\ell)f + f\} \mid f \in H\} + M_\ell(S), \text{directsumin } H^2. \tag{4}$$

This completes the proof.

Theorem 2.2 *If $R(\ell)$ is the corresponding generalized resolvent, then $y = R(\ell)f$ is the unique solution of the boundary value problem*

$$\begin{aligned}
Jy' - H(t)y(t) &= \ell\Delta(t)y(t) + \Delta(t)f(t), \textit{almost all } t \in (a,b), \\
P(\ell)y(a) + D(\ell)y(b) &= 0
\end{aligned} \tag{5}$$

Proof. According to Theorem 1, it is enough to see that the generalized resolvent has the form:

$$\begin{aligned}
R(\ell)f &= G(\ell)f + Y(\ell)\Psi(\ell)[f, Y(\overline{\ell})], \;\; f \in l^2(\Delta dt), & (6) \\
\Psi(\ell) &= -\frac{1}{2}\left(P(\ell) + D(\ell)Y(b,\ell)^{-1}\right)(P(\ell) - D(\ell)Y(b,\ell)J) & (7)
\end{aligned}$$

One can easily check that $\Psi(\ell)^* = \Psi(\overline{\ell})$is an analytic for$\ell \in C^+$, see [16], [20] and [25].

Remark 1 Assume that A is a selfadjoint Hilbert space extension of $T_{\max} \cap Z^*$, which takes place precisely when the kernel $K_\mu(\ell, \lambda)$is non-negative. The Weyl coefficient: $\Psi(\ell)$ as given before is of the Nevanlinna class, which has an integral representation:

$$\Psi(\ell) = A + B\ell + \int_R \left((t-\ell)^{-1} - t(t+1)^{-1}\right) d\Sigma(t) \tag{8}$$

where Σ is monotone nondecreasing function $n \times n$ on $I\mathbb{R}$, and

$$A = A^*, \, \mathrm{Im}B \geq 0, \int_R (t^2+1)^{-1} d\Sigma(t) < \infty$$

where

$$K_\mu(\ell, \lambda) = B + \int_R \frac{d\Sigma(t)}{(t-\ell)(t-\overline{\lambda})},$$

Z is a finite dimensional subspace in H^2,$T_{\min}$ is a minimal relation defined as in [22].

Theorem 2.3 *The Fourier transform$f \to \widehat{f}(\ell) = \left[f, s(\overline{\ell})\right]$ takes $H = L^2_\Delta(a,\, b)$ contractively into L^2_Σ, it is isometric on H_1, has as its kernel H_0 and is strictly contractive on H_z.*

Proof. If A is a minimal selfadjoint extension in $K \supset H$ of $T_{\min} \cap Z^*$ in H, then the Fourier transform is surjective if and only if $\dim K \Theta H = \dim H_z$. The Fourier transform maps H_1 onto L^2_Σ; if and only if A is a canonical selfadjoint extension of $S = T_{\min} \cap Z^*$.

Lemma 2.1 *Let μ, be a matrix function of bounded variation. Define*

$$S = \left\{ \{f,\, g\} \in T_{\min} \mid \int_a^b (d\mu^*)\widetilde{f} = 0 \right\} \tag{9}$$

where f is the unique absolutely continuous representative of f. Then $S = T_{\min} \cap Z^*$, is true if either of the following two cases:

(a) $H(t) = 0$ a.e., in which case $Z =$span$\{-J\mu,\, 0\}, t \in [a,\, b]$

(b) $\Delta(t) = I$ a.e.and $H \in L^2_\Delta(a,\, b)$ in which case $Z =$span$\{-J\mu, HJ\mu\}$.

Proof. We just note $\{f, g\} \in T_{\min}$ implies $Jf' = Hf^{+}\Delta g$, so that integration by parts yields:

$$\int_a^b (d\mu)^* \widetilde{f} = \int_a^b -\mu^* \widetilde{f} = \int (\mu^* J)(J\widetilde{f}) = \int_a^b (\mu^*)(Hf + \Delta g) \tag{10}$$

From this it follows that $S = T_{\min} \cap Z^*$: where $Z = \text{span}\{-J\mu, 0\}$ if $H(t) = 0$, a.e., and where $Z =$span$\{-J\mu, HJ\mu\}$ if $\Delta\,(t)\, I$, a.e., see[17] and[24].

Theorem 2.4 *Let μ and S be defined as above, then $S = T_{\min} \cap Z^*$ is true, if either of the following two cases (assuming that $\Delta\mu$ is defined, see the next remark):*

(a) $H(t) = 0$a.e. in which $Z =$span$\{-\Delta J\mu, 0^n_k\}$,

(b) $H \in L^2_\Delta(a, b)$ in which case $Z =$span$\{-\Delta J\mu, HJ\mu\}$.

Proof. It is easy to check this proof using the previous lemma.

Remark 2 When Δ is singular we do not have such an easy solution. When we multiply Δ y some matrix K, we assume that $R(K) \cap v\Delta = \{(0)\}$, and then all the information is still contained in ΔK, see [15] and [18].

Remark 3 If $s(t)$is defined as before, then one can write

$$\{s(\ell),\ \ell s(\ell)\} = (\{Y(\ell), \ell Y(\ell)\} : \{U(\ell)+\sigma, \ell\,(U(\ell)+\sigma)\}) \tag{11}$$

where $Y(\ell), U(\ell)$ is defined before, $\{\sigma,\ \tau\}$spanned Z, and

$$\begin{aligned} & \{U(\ell)+\sigma, \ell\,(U(\ell)+\sigma)\} && (12) \\ = & \{U(\ell), \ell U(\ell)+\ell\sigma-\tau\} \\ = & \{G(\ell)(-\tau+\ell\sigma), \ell G(\ell)(-\tau+\ell\sigma)-\tau+\ell\sigma\} \\ & +\{Y(\ell), \ell Y(\ell)c(\ell)\}+\{\sigma,\tau\} && (13) \end{aligned}$$

is a decomposition via $T_{\max}+Z$ when $T_{\max}$ is further decomposed as in the above. so, we note that

$$\begin{aligned} s(\ell)(a) &= (I:0) \\ s(\ell)(b) &= \left(Y(b,\ell) : -\frac{1}{2}Y(b,\ell)J[-\tau+\ell\sigma, Y(\bar{\ell})]+Y(b,\ell)c(\ell)\right) \\ & \quad \langle (\{Y(\ell), \ell Y(\ell)\} : \{U(\ell), \ell U(\ell)+\ell\sigma-\tau\}), \{\sigma,\tau\}\rangle \\ &= \left(\left[U(\ell), \bar{\ell}\sigma-\tau\right] : [U(\ell), -\tau+\ell\sigma]+[-\tau+\ell\sigma, \sigma]\right) \\ & \quad \langle \{s(\ell), \ell s(\ell)\}, \{\sigma_0, \tau_0\}\rangle \\ &= (0:I) \end{aligned} \tag{14}$$

hence we obtain

$$b(s(\ell), s(\ell)) = (S_0, S_z) \tag{15}$$

where the matrix S_0, and the matrix S_z are

$$S_0 = \begin{pmatrix} I & 0 \\ Y(b,\ell) & -\frac{1}{2}Y(b,\ell)J[-\tau+\ell\sigma, Y(\bar{\ell})]+Y(b,\ell)c(\ell) \end{pmatrix} \tag{16}$$

$$S_z = \begin{pmatrix} [Y(\ell), \bar{\ell}\sigma-\tau] & [U(\ell), \bar{\ell}\sigma-\tau]+[-\tau+\ell\sigma, \sigma] \\ 0 & I \end{pmatrix} \tag{17}$$

In order to calculate the Weyl coefficient Ψ, we note that for the given self-adjoint extension A, we have on the one hand the following theorem:

Theorem 2.5

$$T(\ell) = \{\{f,\ g\} \in S^* \mid U(\ell)b(\{f,\ g\})\} = 0,$$

where $U(\ell) = (U_0(\ell) : U_z(\ell)$ *: in terms of* Q_0^{-1}, Q_z^{-1}*, but on the other hand,*

$$T(\ell) = \{\{R(\ell)f, (I+\ell R(\ell))f\}|/f \in H\} \tag{18}$$

for the definition of see [19], [21] and [23]

Proof. So for all $f \in H$, we must have:

$$\begin{aligned} & U(\ell)b(\{R(\ell)f, (I+\ell R(\ell))f\}) \\ = & U(\ell)(b(\{G(\ell)f, (\ell G(\ell))f\}) + b(\{s(\ell), \ell s(\ell)\})\Psi(\ell)[f, s(\overline{\ell})]) = 0 \quad (19) \end{aligned}$$

With the usual decomposition $U(\ell) = (U_0(\ell) : Uz(\ell))$, we obtain for all $f \in H$.

$$\begin{aligned} (U_0(\ell)G_0(\ell) + U_z(\ell)G_z(\ell))\,[f, s(\overline{\ell})] + (U_0(\ell)s_0(\ell) + \\ + U_z(\ell)s_z(\ell))\,\Psi(\ell)[f, s(\overline{\ell})] = 0 \end{aligned} \tag{20}$$

Theorem 2.6 *The Weyl coefficient $\Psi(\ell)$ relative to the bounded right inverse $G(\ell)$ is given by:*

$$\Psi(\ell) = -\left(U_0(\ell)s_0(\ell) + U_z(\ell)s_z(\ell)\right)^{-1}\left(U_0(\ell)G_0(\ell) + U_z(\ell)G_z(\ell)\right) \tag{21}$$

where all of these matrices as defined before.

Proof. Just note that for a fixed, the mapping $f \to [f, s(\ell)]$ from H to $C^{2k+2r}$is surjective.

Theorem 2.7 *Let A be a selfadjoint extension of $T_{\min} \cap Z^*$, with a finite dimensional extending space. Then the relation:*

$$T(\ell) = \{\{f, g\} \in T_{\min} + Z \mid U(\ell)b(\{f, g\}) = 0\}\,, \tag{22}$$

also holds for $\ell \in \mathbb{R}$.

In this case the spectrum is discrete, and the eigenvalues, i.e., those values of for which $v[T(\ell) - \ell I] \models \{0\}$, are precisely the values of ℓ, for which the matrix $U_0(\ell)S_0(\ell) + U_z(\ell)S_z(\ell)$ is not invertible, see[8] and[9].

3 Examples

Example(3.1): Consider the second order system in the form

$$-y'' + qy = \ell ry, (q, r \in \mathbb{R}, C, \ell \in \mathbb{C} \backslash \mathbb{R}) \tag{23}$$

which is equivalent to

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} y' \\ qy' - \ell ry \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} - \ell\begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} -q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} + \ell\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix},$$

which can be written in the form $Jz' = \ell\Delta z + Hz$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} -q & 0 \\ 0 & 1 \end{pmatrix}, \Delta = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} y \\ y' \end{pmatrix}$$

In this simple example, one can write the boundary conditions in the form which have been repeated several times in the past, see[2],[3],[4],[5]and[16].

Now we shall consider the general case in a very simple way as one-dimensional vector case, and as we said in the research the coefficients which appear in the differential equations depend on the parameter which is in the lower or in the upper half plane. Let us consider this simple example and construct in each subcase the Nevanlinna function, see[7],[10]and[14].

$$\begin{aligned} iy' - \ell y - \ell\sigma\{P_1 y(0) + Q_1 y(1)\} &= g \\ P_2 y(0) + Q_2 y(1) - i\rho \int_0^1 y d\overline{\sigma} &= 0 \end{aligned} \tag{24}$$

We shall assume that the coefficients P_1, P_2, Q_1 and Q_2 all are real. furthermore, we put $\rho = \ell$ fixed in each case.

Example (3.2): Assume $\rho = 0$. We get this system

$$\begin{aligned} iy^{'} - \ell y &= g \\ P_2 y(0) + Q_2 y(1) &= 0 \end{aligned} \tag{25}$$

We assume the solution of this system (the fundamental solution) in the form:

$$y(t) = c(t)e^{-i\ell t}, c(t)$$

is a constant vector in

$$C^k, \ell \in C^{\pm}, t \in [0, 1].$$

By simple calculation we get the solution and we can construct the Nevanlinna function, from it and even more we may study several cases and in each one construct the Nevanlinna function, see [12], [13] and [25].

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