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Abstract: *The article deals with electrodynamics in the presence of anisotropic materials having scalar wave impedance. Maxwell's equations written for differential forms over a 3-manifold are analysed. The system is extended to a Dirac type first order elliptic system on the Grassmannian bundle over the manifold. The second part of the article deals with the dynamical inverse boundary value problem of determining the electromagnetic material parameters from boundary measurements. By using the boundary control method, it is proved that the dynamical boundary data determines the electromagnetic travel time metric as well as the scalar wave impedance on the manifold. This invariant result leads also to a complete characterization of the non-uniqueness of the corresponding inverse problem in bounded domains of* $\mathbb{R}^3$.

AMS subject classifications: 35A21, 35J55, 35L50, 35Q60, 35R30, 53C21, 58A10, 58J32, 58J90, 78A25

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Introduction

Classically, the laws of electromagnetism expressed by Maxwell's equations are written for vector fields representing the electric and magnetic fields. However, it is possible to rephrase these equations in terms of differential forms. It turns out that this alternative formulation has several advantages both from the theoretical and practical point of view. First, the formulation of electromagnetics with differential forms reflect the way in which the fields are actually observed. For instance, flux quantities are expressed as 2–forms while field quantities that correspond to forces are naturally written as 1–forms. This point of view has been adopted in modern physics at least when fields in free space are dealt with, see [18]. Furthermore, the formulation distinguishes the topological properties of the electromagnetic media from those that depend on geometry. It is understood that geometry is related to the properties of the material where the waves propagate. The distinction between non-geometric and geometric properties has consequences also to the numerical treatment of the equations by so called Whitney forms. An extensive treatment of this topic can be found in [12], [13]. For the original reference concerning Whitney elements see [58].

The present work is divided in two parts. In the first part, we pursue further the invariant formulation of Maxwell's equations to model the wave propagation in certain anisotropic materials. More precisely, we consider anisotropic materials with scalar wave impedance. Physically, scalar wave impedance is tantamount to a single propagation speed of waves with different polarization. The invariant approach leads us to formulate Maxwell's equations on 3-manifolds as a first order Dirac type system. From the operator theoretic point of view, this formulation is based on an elliptization procedure by extending Maxwell's equations to a Grassmannian bundle over the manifold. This is a generalization of the elliptization of Birman and Solomyak and Picard (see[1],[46]).

In the second part of the work, we consider the inverse boundary value problem for Maxwell's equations. In terms of physics, the goal is to determine material parameter tensors, electric permittivity ϵ and magnetic permeability μ, in a bounded domain from field observations at the boundary of that domain. As it is already well established, for anisotropic inverse problems it is natural to consider the problem in two parts. First, we consider the invariant problem on a Riemannian manifold, where we recover the travel time metric and the wave impedance on the manifold. As a second step, we consider the consequences of the invariant result when the manifold is imbedded to $\mathbb{R}^3$.

Although inverse problems in electrodynamics have a great significance in physics and applications, results concerning the multidimensional inverse problems are relatively recent. One-dimensional results have existed starting from the 30'ies, see e.g. [34], [50]. The first breakthrough in multidimensional inverse problems for electrodynamics was based on the use of complex geometrical optics [52], [15], [43], [44]. In these papers, the inverse problem of recovering the scalar material parameters from complete fixed frequency boundary data was solved even in the non-selfadjoint case, i.e., in the presence of electric conductivity. These works were based on ideas previously developed in references [54],[39],[40] to solve the scalar Calderón problem, that obtained its present formulation in [14].

In the dynamical case, a method to solve an isotropic inverse boundary problem based on ideas of integral geometry is developed in [48]. The method, however, is confined to the case of a geodesically simple manifolds and, at the moment, is limited to finding some combinations of material parameters, including electric conductivity. An alternative method to tackle the inverse boundary value problem is the boundary control (BC) method, originated in [4]. Later, this method was developed for the Laplacian on Riemannian manifolds [7] and for anisotropic self-adjoint [27] – [29] and certain non-selfadjoint inverse problems [32]. The first application of the BC method to electrodynamics was done in [9], [6]. The authors of these articles show that, when the material parameters ϵ and μ are real scalars or alternatively when $\epsilon = \mu$, the boundary data determines the wave speed in the vicinity of the boundary. These works employed the Hodge-Weyl decomposition in the domain of influence near the boundary. The real obstruction for this technique is that, as time grows, the domain of influence can become non-smooth and the topology may be highly involved. For these reasons, our paper is based on different ideas.

In this article, there are essentially two new leading ideas. First, we characterize the subspaces controlled from the boundary by duality, thus avoiding the difficulties arising from the complicated topology of the domain of influence. The second idea is to develop a method of waves focusing at a single point of the manifold. This enables us to recover pointwise values of the waves on the manifold. The geometric techniques of the paper are presented in [30] and the book [25].

The main results of this paper can be summarized as follows.

1. The knowledge of the complete dynamical boundary data over a sufficiently large finite period of time determines uniquely the compact manifold endowed with the electromagnetic travel time metric as well as the scalar wave impedance (Theorem 4.1).

2. For the corresponding anisotropic inverse boundary value problem with scalar wave impedance for bounded domains in $\mathbb{R}^3$, the non-uniqueness is completely characterized by describing the class of possible transformations between material tensors that are indistinguishable from the boundary (Theorem 11.1).

To the best knowledge of the authors, no global uniqueness results for inverse problems for systems with anisotropic coefficients have been previously known.

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1 Maxwell's equations for forms

In this chapter we derive an invariant form for Maxwell equations, consider initial boundary value problem for them and show how energy of fields can be found using boundary measurements.

We start with Maxwell equations in domain $\Omega \subset \mathbb{R}^3$ equipped with the standard Euclidean structure. Since our objective is to write Maxwell equations in an invariant form, we generalize the setting in very beginning and instead of domain Ω consider manifolds.

Let (M, g_0) be a connected, oriented Riemannian 3-manifold possibly with a boundary $\partial M \neq \emptyset$. We assume that all objects in this paper are C^∞–smooth. Consider Maxwell's equations on M,

$$\operatorname{curl} E = -B_t, \quad \text{(Maxwell–Faraday)}, \tag{1}$$

$$\operatorname{curl} H = D_t, \quad \text{(Maxwell–Ampère)}, \tag{2}$$

where E and H are the electric and magnetic fields, and B and D are the magnetic flux density and electric displacement, assumed for the time being to be smooth mappings $M \times \mathbb{R} \to TM$. Here TM denotes the tangent bundle over M. The curl operator as well as divergence appearing later will be defined invariantly in formula (5) below. The sub-index t in the equations (1)–(2) denotes differentiation with respect to time. We denote the collection of these vector fields as $\Gamma(M \times \mathbb{R})$. At this point, we do not specify the initial and boundary values. To avoid non-physical static solutions, the above equations are augmented with the conditions

$$\operatorname{div} B = 0, \quad \operatorname{div} D = 0. \tag{3}$$

Furthermore, the fields E and D, and similarly the fields H and B are interrelated through the constitutive relations. In anisotropic and non-dispersive medium, the constitutive relations assume the simple form

$$D = \epsilon E, \quad B = \mu H, \tag{4}$$

where ϵ, μ are smooth and strictly positive definite tensor fields of type $(1, 1)$ on M. Our aim is to write the above equations using differential forms.

Given the metric g_0, we can associate in a canonical way a differential 1–form to correspond each vector field. Let us denote by $\wedge^k T^*M$ the k:th exterior power of the cotangent bundle. We define the mapping

$$TM \to T^*M, \quad X \mapsto X^\flat$$

through the formula $g_0(X,Y) = X^\flat(Y)$. This mapping is one-to-one and it has the following well-known properties (See e.g. [51]): For a scalar field $u \in C^\infty(M)$, $(\operatorname{grad} u)^\flat = du$, where d is the exterior differential and for a vector field $X \in \Gamma(M)$, we have

$$(\operatorname{curl} X)^\flat = *_0 dX^\flat, \quad \operatorname{div} X = -\delta_0 X^\flat, \tag{5}$$

where $*_0$ denotes the Hodge–$*$ operator with respect to the metric g_0,

$$*_0 : \wedge^k T^*M \to \wedge^{3-k} T^*M,$$

and δ_0 denotes the codifferential [1],

$$\delta_0 = (-1)^k *_0 d *_0 : \Omega^k M \to \Omega^{k-1} M.$$

Here, $\Omega^k M$ denotes the smooth sections $M \to \wedge^k T^*M$, i.e. differential $k-$forms. Applying now the operator $\flat$ on Maxwell's equations (1)–(2) yields

$$dE^\flat = - *_0 B_t^\flat, \quad dH^\flat = *_0 D_t^\flat,$$

where we used the identity $*_0 *_0 = \mathrm{id}$ valid in 3–geometry[2]. The divergence equations (3) read

$$\delta_0 D^\flat = 0, \quad \delta_0 B^\flat = 0.$$

Consider now the constitutive relations (4). Starting with the equation $D = \epsilon E$, we pose the following question: Is it possible to find a *metric* g_ϵ such that the Hodge-$*$ operator with respect to this metric, denoted by $*_\epsilon$, would satisfy the identity

$$*_0 D^\flat = *_0 (\epsilon E)^\flat = *_\epsilon E^\flat?$$

Assume that such a metric g_ϵ exists. By writing out the above formula in given local coordinates (x^1, x^2, x^3) and recalling the definition of the Hodge-$*$ operator, the left side yields

$$\begin{aligned} *_0(\epsilon E)^\flat &= \sqrt{\mathrm{g}_0}\, g_0^{ij} e_{jpq} g_{0,ij} \epsilon^j_k E^k dx^p \wedge dx^q \\ &= \sqrt{\mathrm{g}_0}\, e_{jpq} \epsilon^j_k E^k dx^p \wedge dx^q, \end{aligned}$$

where e is the totally antisymmetric permutation index and $\mathrm{g}_0 = \det(g_{0,ij})$. Likewise, the right side reads

$$*_\epsilon E^\flat = \sqrt{\mathrm{g}_\epsilon}\, g_\epsilon^{ij} e_{jpq} g_{0,ik} E^k dx^p \wedge dx^q,$$

[1] Cf. with $\delta_0 = (-1)^{nk+n+1} *_0 d *_0$ for Riemannian n–manifolds

[2] For Riemannian n–manifold, we have in general $*_0 *_0 = (-1)^{k(n-k)}$

so evidently the desired equality ensues if we set

$$\sqrt{\mathrm{g}_\epsilon} g_\epsilon^{ij} g_{0,ik} = \sqrt{\mathrm{g}_0} \epsilon_j^k.$$

By taking determinants of both sides we find that

$$\sqrt{\mathrm{g}_\epsilon} = \sqrt{\mathrm{g}_0} \det(\epsilon).$$

Thus we see that the appropriate form for the metric tensor in the contravariant form is

$$g_\epsilon^{ij} = \frac{1}{\det(\epsilon)} g_0^{ik} \epsilon_k^j. \tag{6}$$

In the same fashion, we find a metric g_μ such that

$$*_0 B^\flat = *_0 (\mu H)^\flat = *_\mu H^\flat.$$

In general, the metrics g_μ and g_ϵ can be very different from each other. In this article, we consider a particular case. Indeed, assume that the material has a *scalar wave impedance*, i.e., the tensors ϵ and μ satisfy

$$\mu = \alpha^2 \epsilon,$$

where the wave impedance, $\alpha = \alpha(x)$, is a smooth function on M. Now we define two families of 1– and 2–forms on M as follows. We set

$$\omega^1 = E^\flat, \quad \omega^2 = *_0 B^\flat.$$

Similarly, we define

$$\nu^1 = \alpha H^\flat, \quad \nu^2 = *_0 \alpha D^\flat. \tag{7}$$

Observe that the wave impedance scaling renders ω^1 and η^1 to have the same physical dimensions, and the same holds for the 2–form. Now it is a straightforward matter to check that the constitutive relations assume the form

$$\nu^2 = \alpha *_\epsilon \omega^1, \quad \omega^2 = \frac{1}{\alpha} *_\mu \nu^1.$$

We can make these equations even more symmetric by proper scaling of the metrics. Indeed, since $\alpha^{-1}\mu = \alpha\epsilon$, we have a new metric g that is defined as

$$g^{ij} = g_{\alpha\epsilon}^{ij} = g_{\alpha^{-1}\mu}^{ij}.$$

We have, by direct substitution that

$$g^{ij} = \frac{1}{\alpha^2} g_\epsilon^{ij} = \alpha^2 g_\mu^{ij}. \tag{8}$$

This new metric will be called the *travel time metric* in the sequel.

Assume that

$$* : \wedge^j T^*M \to \wedge^{3-j} T^*M$$

denotes the Hodge–$*$ operator with respect to some metric $\widehat{g}$. If we perform a scaling of the metric as

$$\widehat{g}^{ij} \to \widetilde{g}^{ij} = r^2 \widehat{g}^{ij},$$

the corresponding Hodge operator is scaled as

$$* \to \widetilde{*} = r^{2j-3} * .$$

Therefore, if we denote by $*$ the Hodge–$*$ operator with respect to the travel time metric, we have

$$* = \alpha *_\epsilon = \frac{1}{\alpha} *_\mu : \wedge^1 T^*M \to \wedge^2 T^*M.$$

But this means simply that, in terms of the travel time metric, we have

$$\nu^2 = *\omega^1, \quad \omega^2 = *\nu^1. \tag{9}$$

Consider now Maxwell's equations for these forms. After eliminating the ν–forms using the constitutive equations (9), Maxwell–Faraday and Maxwell–Ampère equations assume the form

$$d\omega^1 = -\omega^2_t, \quad \delta_\alpha \omega^2 = \omega^1_t, \quad \delta_\alpha = (-1)^k * \alpha d \frac{1}{\alpha} * : \Omega^k M \to \Omega^{k-1} M \tag{10}$$

and the divergence equations (3) read

$$d\omega^2 = 0, \quad \delta_\alpha \omega^1 = 0. \tag{11}$$

In the sequel, we call equations (10) and (11) *Maxwell's equations.*

It turns out to be useful to define auxiliary forms that vanish in the electromagnetic theory. Let us introduce the auxiliary forms ω^0 and ω^3 via the formulas

$$\omega^0_t = \delta_\alpha \omega^1, \quad -\omega^3_t = d\omega^2.$$

Furthermore, we define the corresponding ν–forms as

$$\nu^0 = *\omega^3, \quad \nu^3 = *\omega^0. \tag{12}$$

Since these auxiliary forms are all vanishing, we may modify the equations (10) to have

$$d\omega^1 - \delta_\alpha \omega^3 = -\omega^2_t, \quad d\omega_0 - \delta_\alpha \omega^2 = -\omega^1_t. \tag{13}$$

Putting the obtained equations together in a matrix form, we arrive at the equation

$$\omega_t + \mathcal{M}\omega = 0, \tag{14}$$

where

$$\omega = (\omega^0, \omega^1, \omega^2, \omega^3)$$

and the operator $\mathcal{M}$ (without defining its domain at this point, i.e., defined as a differential expression) is given as

$$\mathcal{M} = \begin{pmatrix} 0 & -\delta_\alpha & 0 & 0 \\ d & 0 & -\delta_\alpha & 0 \\ 0 & d & 0 & -\delta_\alpha \\ 0 & 0 & d & 0 \end{pmatrix}. \tag{15}$$

The equation (14) is called *the complete Maxwell system*. In the next section, we treat more systematically this operator.

Remark 1. The operator $\mathcal{M}$ has the property

$$\mathcal{M}^2 = -\text{diag}(\Delta_\alpha^0, \Delta_\alpha^1, \Delta_\alpha^2, \Delta_\alpha^3) = -\boldsymbol{\Delta}_\alpha,$$

where the operator Δ_α^k acting on k–forms is

$$\Delta_\alpha^k = d\delta_\alpha + \delta_\alpha d = \Delta_g^k + Q(x, D),$$

with Δ_g^k denoting the Laplace-Beltrami operator on k–forms with respect to the travel time metric and $Q(x, D)$ being a first order perturbation. Hence, if ω satisfies the equation (14), we have

$$(\partial_t^2 + \boldsymbol{\Delta}_\alpha)\omega = (\partial_t - \mathcal{M})(\partial_t + \mathcal{M})\omega = 0.$$

In particular, we observe that the assumption that the impedance is scalar implies a unique propagation speed for the system.

Remark 2. Denote by $\Omega M = \oplus_{k=0}^3 \Omega^k M$ the Grassmannian algebra of differential forms, where $\Omega^k M$ are the differential k-forms. Then the operator $\mathcal{M}$ in formula (15) can be also considered as a Dirac operator $d - \delta_\alpha : \Omega M \to \Omega M$.

Before leaving this section, let us briefly consider the energy integrals in terms of the differential forms. In terms of the vector fields, the energy of the electric field at a given moment t is obtained as the integral

$$\mathcal{E}(E) = \int_M \epsilon E \cdot E dV = \int_M g_0(E, D) dV = \int_M E^\flat \wedge *_0 D^\flat$$

where dV is volume form of (M, g_0). By plugging in the defined forms we arrive at

$$\mathcal{E}(E) = \int_M \frac{1}{\alpha} \omega^1 \wedge *\omega^1.$$

In the same fashion, we find that the energy of the magnetic field reads

$$\mathcal{E}(B) = \int_M \frac{1}{\alpha} \omega^2 \wedge *\omega^2.$$

These formulas serve as a motivation for our definition of the inner product in the following section.

1.1 Maxwell operator

In this section we establish a number of notational conventions and definitions concerning the differential forms used in this work.

We define the L^2–inner products for k–forms in $\Omega^k M$ as

$$(\omega^k,\eta^k)_{L^2} = \int_M \frac{1}{\alpha}\omega^k\wedge *\eta^k, \quad \omega^k,\ \eta^k\in\Omega^k M.$$

Further, we denote by $L^2(\Omega^k M)$ the completion of $\Omega^k M$ with respect to the norm defined by the above inner products. We also define

$$\mathbf{L}^2(M) = L^2(\Omega^0 M)\times L^2(\Omega^1 M)\times L^2(\Omega^2 M)\times L^2(\Omega^3 M).$$

Similarly, we define Sobolev spaces $\mathbf{H}^s(M)$, $s\in\mathbb{R}$,

$$\mathbf{H}^s(M) = H^s(\Omega^0 M)\times H^s(\Omega^1 M)\times H^s(\Omega^2 M)\times H^s(\Omega^3 M),$$

$$\mathbf{H}^s_0(M) = H^s_0(\Omega^0 M)\times H^s_0(\Omega^1 M)\times H^s_0(\Omega^2 M)\times H^s_0(\Omega^3 M),$$

where $H^s(\Omega^k M)$ are Sobolev spaces of $k-$ forms. At last, $H^s_0(\Omega^k M)$ is the closure in $H^s(\Omega^k M)$ of $\Omega^k M^{\rm int}$, i.e. the subspace of $\Omega^k M$ of $k-$ forms which vanish near ∂M.

The domain of the exterior derivative d in the L^2–space of k–forms is

$$H(d,\Omega^k M) = \left\{\omega^k\in L^2(\Omega^k M)\mid d\omega^k\in L^2(\Omega^{k+1}M)\right\}.$$

Similarly, we set

$$H(\delta_\alpha,\Omega^k M) = \left\{\omega^k\in L^2(\Omega^k M)\mid \delta_\alpha\omega^k\in L^2(\Omega^{k-1}M)\right\},$$

where δ_α is the weak extension of the operator $\delta_\alpha : \Omega^k M\to\Omega^{k-1}M$. In the sequel, we shall drop the sub-index α from the codifferential.

The codifferentiation δ is adjoint to the exterior derivative in the sense that for C_0^∞–forms on M,

$$(d\omega^k,\eta^{k+1})_{L^2} = (\omega^k,\delta\eta^{k+1})_{L^2}.$$

To extend the adjoint formula for less regular forms, let us first fix some notations. For $\omega^k\in\Omega^k M$, we define the *tangential* and *normal* boundary data at ∂M as

$$\mathbf{t}\omega^k = i^*\omega^k, \quad \mathbf{n}\omega^k = i^*(\frac{1}{\alpha}*\omega^k),$$

respectively, where $i^*:\Omega^k M\to\Omega^k\partial M$ is the pull-back of the natural imbedding $i:\partial M\to M$. Sometimes, we denote $\mathbf{n}=\mathbf{n}_\alpha$ to indicate a particular choice α. With these notations, let us write

$$\int_{\partial M} i^*\omega^k\wedge i^*(\frac{1}{\alpha}*\eta^{k+1}) = \langle\mathbf{t}\omega^k,\mathbf{n}\eta^{k+1}\rangle.$$

We add here a small *caveat* that the above formula does not define an inner product as ω^k and η^{k+1} are differential forms of different order. For $\omega^k \in \Omega^k M$ and $\eta^{k+1} \in \Omega^{k+1} M$, the Stokes formula for forms can be written as

$$(d\omega^k, \eta^{k+1})_{L^2} - (\omega^k, \delta\eta^{k+1})_{L^2} = \langle \mathbf{t}\omega^k, \mathbf{n}\eta^{k+1} \rangle. \tag{16}$$

This formula allows the extension of the boundary trace operators $\mathbf{t}$ and $\mathbf{n}$ to $H(d, \Omega^k M)$ and $H(\delta, \Omega^k M)$, respectively. Indeed, if $\omega^k \in H^1(\Omega^k M)$, then $\mathbf{t}\omega^k \in H^{1/2}(\Omega^k \partial M)$ and, by formula (16), we may extend

$$\mathbf{t} : H(d, \Omega^k M) \to H^{-1/2}(\Omega^k \partial M).$$

In the same way, equation (16) gives us the natural extension

$$\mathbf{n} : H(\delta, \Omega^{k+1} M) = H^{-1/2}(\Omega^{2-k} \partial M),$$

In fact, a stronger result holds.

Proposition 1.1 *The operators* $\mathbf{t}$ *and* $\mathbf{n}$ *can be extended to continuous surjective maps*

$$\begin{aligned} \mathbf{t} : H(d, \Omega^k M) &\to H^{-1/2}(d, \Omega^k \partial M), \\ \mathbf{n} : H(\delta, \Omega^{k+1} M) &\to H^{-1/2}(d, \Omega^{2-k} \partial M), \end{aligned}$$

where the space $H^{-1/2}(d, \Omega^k \partial M)$ *is the space of* k*-forms* ω^k *on* ∂M *satisfying*

$$\omega^k \in H^{-1/2}(\Omega^k \partial M), \quad d\omega^k \in H^{-1/2}(\Omega^{k+1} \partial M).$$

This result is due to Paquet [45].

The formula (16) can be used also to define function spaces with vanishing boundary data. Indeed, let us define

$$\begin{aligned} \overset{\circ}{H}(d, \Omega^k M) = \{\omega^k \in H(d, \Omega^k M) \ | \ & (d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \delta\eta^{k+1})_{L^2} \\ & \text{for all } \eta^{k+1} \in H(\delta, \Omega^{k+1} M)\}, \\ \overset{\circ}{H}(\delta, \Omega^{k+1} M) = \{\eta^{k+1} \in H(\delta, \Omega^{k+1} M) \ | \ & (d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \delta\eta^{k+1})_{L^2} \\ & \text{for all } \omega^k \in H(d, \Omega^k M)\}. \end{aligned}$$

It is not hard to see that indeed

$$\overset{\circ}{H}(d, \Omega^k M) = \mathbf{t}^{-1}\{0\}, \quad \overset{\circ}{H}(\delta, \Omega^{k+1} M) = \mathbf{n}^{-1}\{0\}.$$

We are now in the position prove the following lemma.

Lemma 1.2 *The adjoint of the operator*

$$d : L^2(\Omega^k M) \supset H(d, \Omega^k M) \to L^2(\Omega^{k+1} M)$$

is the operator

$$\delta : L^2(\Omega^{k+1}M) \supset \overset{\circ}{H}(\delta, \Omega^{k+1}M) \to L^2(\Omega^k M)$$

and vice versa. *Similarly, the adjoint of*

$$\delta : L^2(\Omega^{k+1}M) \supset H(\delta, \Omega^{k+1}M) \to L^2(\Omega^k M)$$

is the operator

$$d : L^2(\Omega^k M) \supset \overset{\circ}{H}(d, \Omega^k M) \to L^2(\Omega^{k+1}M)$$

Proof: We prove only the first claim, the other having a similar proof.

Let $\eta^{k+1} \in \mathcal{D}(d^*)$, where d^* denotes the adjoint of d. By definition, there exists $\vartheta^k \in L^2(\Omega^k M)$ such that

$$(d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \vartheta^k)_{L^2}$$

for all $\omega^k \in H(d, \Omega^k M)$. In particular, if $\omega^k \in \Omega^k M^{\rm int}$, we see that, in the weak sense,

$$(d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \delta\eta^{k+1})_{L^2} = (\omega^k, \vartheta^k)_{L^2},$$

i.e., $\delta\eta^{k+1} = \vartheta^k \in L^2(\Omega^k M)$. Thus, $\eta^{k+1} \in H(\delta, \Omega^{k+1}M)$, and the claim follows now since we have

$$(d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \delta\eta^{k+1})_{L^2}$$

for all $\omega^k \in H(d, \Omega^k M)$, i.e., $\delta = d^*$. □

In the sequel, we will write for brevity $H(d) = H(d, \Omega^k M)$, etc. when there is no risk of confusion concerning the order of the forms.

For later reference, let us point out that the Stokes formula for the complete Maxwell system can be written compactly as

$$(\eta, \mathcal{M}\omega)_{\mathbf{L}^2} + (\mathcal{M}\eta, \omega)_{\mathbf{L}^2} = \langle \mathbf{t}\omega, \mathbf{n}\eta \rangle + \langle \mathbf{t}\eta, \mathbf{n}\omega \rangle, \tag{17}$$

where $\omega \in \mathbf{H}$ with

$$\mathbf{H} = H(d) \times [H(d) \cap H(\delta)] \times [H(d) \cap H(\delta)] \times H(\delta) \tag{18}$$

and $\eta \in \mathbf{H}^1(M)$ and we use the notations

$$\mathbf{t}\omega = (\mathbf{t}\omega^0, \mathbf{t}\omega^1, \mathbf{t}\omega^2) \qquad \mathbf{n}\omega = (\mathbf{n}\omega^3, \mathbf{n}\omega^2, \mathbf{n}\omega^1),$$

and, naturally,

$$\langle \mathbf{t}\omega, \mathbf{n}\eta \rangle = \langle \mathbf{t}\omega^0, \mathbf{n}\eta^1 \rangle + \langle \mathbf{t}\omega^1, \mathbf{n}\eta^2 \rangle + \langle \mathbf{t}\omega^2, \mathbf{n}\eta^3 \rangle.$$

With these notations, we give the following definition of the Maxwell operators with electric and magnetic boundary conditions, respectively.

Definition 1.3 *The Maxwell operator with the electric boundary condition, denoted by*

$$\mathcal{M}_{\mathrm{e}} : \mathcal{D}(\mathcal{M}_{\mathrm{e}}) \to \mathbf{L}^2(M),$$

is defined through the differential expression (15), with the domain $\mathcal{D}(\mathcal{M}_{\mathrm{e}}) \subset \mathbf{L}^2(M)$ *defined as*

$$\mathcal{D}(\mathcal{M}_{\mathrm{e}}) = \overset{\circ}{\mathbf{H}}_{\mathbf{t}} := \overset{\circ}{H}(d) \times [\overset{\circ}{H}(d) \cap H(\delta)] \times [\overset{\circ}{H}(d) \cap H(\delta)] \times H(\delta).$$

Similarly, the Maxwell operator with the magnetic boundary condition, denoted **by**

$$\mathcal{M}_{\mathrm{m}} : \mathcal{D}(\mathcal{M}_{\mathrm{m}}) \to \mathbf{L}^2(M),$$

is defined through the differential expression (15), with the domain $\mathcal{D}(\mathcal{M}_{\mathrm{m}}) \subset \mathbf{L}^2(M)$ *defined as*

$$\mathcal{D}(\mathcal{M}_{\mathrm{m}}) = \overset{\circ}{\mathbf{H}}_{\mathbf{n}} := H(d) \times [H(d) \cap \overset{\circ}{H}(\delta)] \times [H(d) \cap \overset{\circ}{H}(\delta)] \times \overset{\circ}{H}(\delta).$$

Before further discussion, let us comment the boundary conditions in terms of physics. For vectorial representations of the electric and magnetic fields, the electric boundary condition is associated with electrically perfectly conducting boundaries, i.e., $n \times E = 0$, $n \cdot B = 0$, where n is the exterior normal vector at the boundary. In terms of differential forms, this means simply that $\mathbf{t}E^\flat = \mathbf{t}\omega^1 = 0$ and $\mathbf{t} *_0 B^\flat = \mathbf{t}\omega^2 = 0$. On the other hand, the magnetic boundary conditions represent a magnetically perfectly conducting boundaries, i.e., $n \times H = 0$, $n \cdot D = 0$, which again in terms of forms reads as $\mathbf{t}H^\flat = \mathbf{t}(1/\alpha)\nu^1 = 0$ or $\mathbf{t}(1/\alpha) * \omega^2 = \mathbf{n}\omega^2 = 0$ and $\mathbf{t} *_0 D^\flat = \mathbf{t}(1/\alpha)\nu^2 = 0$, or in terms of ω^1, $\mathbf{t}(1/\alpha) * \omega^1 = \mathbf{n}\omega^1 = 0$.

There is an obvious duality between these conditions. It is therefore sufficient to consider the operator with the electric boundary condition only. This observation is related to the well-known Maxwell *duality principle.*

Consider the intersections of spaces appearing in the domains of definition in the previous definition. Let us denote

$$\overset{\circ}{H}{}^1_{\mathbf{t}}(\Omega^k M) = \{\omega^k \in H^1(\Omega^k M) \mid \mathbf{t}\omega^k = 0\},$$
$$\overset{\circ}{H}{}^1_{\mathbf{n}}(\Omega^k M) = \{\omega^k \in H^1(\Omega^k M) \mid \mathbf{n}\omega^k = 0\}.$$

It is a direct consequence of Gaffney's inequality (see [51]) that

$$\overset{\circ}{H}(d, \Omega^k M) \cap H(\delta, \Omega^k M) = \overset{\circ}{H}{}^1_{\mathbf{t}}(\Omega^k M),$$
$$H(d, \Omega^k M) \cap \overset{\circ}{H}(\delta, \Omega^k M) = \overset{\circ}{H}{}^1_{\mathbf{n}}(\Omega^k M).$$

The following lemma is a direct consequence of Lemma 1.2 and classical results on Hodge-Weyl decomposition[51].

Lemma 1.4 *The electric Maxwell operator has the following properties:*

i. *The operator* $\mathcal{M}_e$ *is skew-adjoint.*

ii. *The operator* $\mathcal{M}_e$ *defines an elliptic differential operator in* M^{int}.

iii. $Ker(\mathcal{M}_e) = \{(0, \omega^1, \omega^2, \omega^3) \in \overset{\circ}{\mathbf{H}}_{\mathbf{n}} : \ d\omega^1 = 0, \ \delta\omega^1 = 0, \ d\omega^2 = 0, \ \delta\omega^2 = 0, \ \delta\omega^3 = 0\}$.

iv. $Ran(\mathcal{M}_e) = L^2(\Omega^0 M) \times (\delta H(\delta, \Omega^2 M) + d\overset{\circ}{H}(d, \Omega^0 M)) \times (\delta H(\delta, \Omega^2 M) + d\overset{\circ}{H}(d, \Omega^1 M)) \times d\overset{\circ}{H}(d, \Omega^2 M)$.

By the skew-adjointness, it is possible to define weak solutions to initial-boundary-value problems needed later. In the sequel we denote the forms $\omega(x, t)$ just by $\omega(t)$ when there is no danger of misunderstanding.

Definition 1.5 *By the* weak solution *to the initial boundary value problem*

$$\omega_t + \mathcal{M}\omega = \rho \in L^1_{loc}(\mathbb{R}, \mathbf{L}^2(M)),$$

$$\mathbf{t}\omega|_{\partial M \times \mathbb{R}} = 0, \quad \omega(\,\cdot\,, 0) = \omega_0 \in \mathbf{L}^2, \tag{19}$$

we mean the form

$$\omega(t) = \mathcal{U}(t)\omega_0 + \int_0^t \mathcal{U}(t-s)\rho(s)ds,$$

where $\mathcal{U}(t) = \exp(-t\mathcal{M}_e)$ *is the unitary operator generated by* $\mathcal{M}_e$.

In the analogous manner, we define weak solutions with initial data given on $t = T$, $T \in \mathbb{R}$. Assuming $\rho \in C(\mathbb{R}, \mathbf{L}^2(M))$ and using the theory of unitary groups, we immediately obtain the regularity result

$$\omega \in C(\mathbb{R}, \mathbf{L}^2) \cap C^1(\mathbb{R}, \mathbf{H}').$$

where $\mathbf{H}'$ denotes the dual of $\mathbf{H}$.

We shall need later the boundary traces of the weak solution. To define them, let $(\omega_{0n}, \rho_n) \in \mathcal{D}(\mathcal{M}_e) \times C(\mathbb{R}, \mathcal{D}(\mathcal{M}_e))$ be an approximating sequence of the pair (ω_0, ρ) in $\mathbf{L}^2 \times C(\mathbb{R}, \mathbf{L}^2)$. We define

$$\omega_n = \mathcal{U}(t)\omega_{0n} + \int_0^t \mathcal{U}(t-s)\rho_n(s)ds,$$

whence $\omega_n \in C(\mathbb{R}, \mathcal{D}(\mathcal{M}_e)) \cap C^1(\mathbb{R}, \mathbf{L}^2)$. Let $\varphi = (\varphi^0, \varphi^1, \varphi^2)$ be a test form, $\varphi^j \in C_0^\infty(\mathbb{R}, \Omega^j \partial M)$. Let η be a strong solution of the initial boundary value problem

$$\eta_t + \mathcal{M}\eta = 0,$$

$$\mathbf{t}\eta = \varphi, \quad \eta(\,\cdot\,,0) = 0.$$

We have

$$\begin{aligned}(\eta(T), \omega_n(T))_{\mathbf{L}^2} &= \int_0^T \partial_t(\eta, \omega_n)dt \\ &= -\int_0^T ((\mathcal{M}\eta, \omega_n) + (\eta, \mathcal{M}\omega_n))dt,\end{aligned}$$

and, by applying Stokes theorem, we deduce

$$(\eta(T), \omega_n(T))_{\mathbf{L}^2} = -\int_0^T \langle \varphi, \mathbf{n}\omega_n \rangle.$$

Hence, we observe that, when going to the limit $n \to \infty$, the above formula defines $\mathbf{n}\omega = \lim_{n\to\infty} \mathbf{n}\omega_n \in \mathcal{D}'(\mathbb{R}, \mathcal{D}'(\partial M))$, where

$$\mathcal{D}'(\partial M) = \mathcal{D}'(\Omega^0 \partial M) \times \mathcal{D}'(\Omega^1 \partial M) \times \mathcal{D}'(\Omega^2 \partial M).$$

We conclude this section with the following result.

Lemma 1.6 *Assume that the initial data ω_0 is of the form $\omega_0 = (0, \omega_0^1, \omega_0^2, 0)$, where $\delta\omega_0^1 = 0$, $d\omega_0^2 = 0$ and we have $\rho = 0$. Then the weak solution ω of Definition 1.5 satisfies also Maxwell's equations (10), (11), i.e., $\omega^0 = 0$ and $\omega^3 = 0$.*

Proof: As observed in Remark 1, ω and, in particular, ω^0 satisfies the wave equation

$$\Delta_\alpha^0 \omega^0 + \omega_{tt}^0 = 0,$$

in the distributional sense, along with the Dirichlet boundary condition $\mathbf{t}\omega^0 = 0$. The initial data for ω^0 is

$$\omega^0(0) = \omega_0^0 = 0,$$

and

$$\omega_t^0(0) = \delta\omega^1|_{t=0} = \delta\omega_0^1 = 0.$$

Hence, we deduce that also $\omega^0 = 0$.

Similarly, ω^3 satisfies the wave equation with the initial data

$$\omega^3(0) = \omega_3^0 = 0,$$

and

$$\omega_t^3(0) = -d\omega^2|_{t=0} = -d\omega_0^2 = 0.$$

As for the boundary condition, we observe that

$$\mathbf{t}\delta\omega^3 = \mathbf{t}\omega_t^2 - \mathbf{t}d\omega^1 = \partial_t \mathbf{t}\omega^2 - d\mathbf{t}\omega^1 = 0,$$

corresponding to the vanishing Neumann data for the function $*\omega^3$. Thus, also $\omega^3 = 0$. □

1.2 Initial–boundary value problem

Our next goal is to consider the forward problem and the Cauchy data on the lateral boundary $\partial M \times \mathbb{R}$ for solutions of Maxwell's equations. Assume that ω is a solution of the complete system. The complete Cauchy data of this solution consists of

$$(\mathbf{t}\omega(x,t), \mathbf{n}\omega(x,t)), \quad (x,t) \in \partial M \times \mathbb{R}_+.$$

Assume now that ω corresponds to the solution of Maxwell's equations, i.e., we have $\omega^0 = 0$ and $\omega^3 = 0$. Consider the Maxwell-Faraday equation in (10),

$$\omega_t^2 + d\omega^1 = 0.$$

By taking the tangential trace, we find that $\mathbf{t}\omega_t^2 = -d\mathbf{t}\omega^1$, and further,

$$\mathbf{t}\omega^2(x,t) = \omega^2(0) - \int_0^t d\mathbf{t}\omega^1(x,t')dt', \quad x \in \partial M.$$

Similarly, by taking the normal trace of the Maxwell-Ampère equation in (10),

$$\omega_t^1 - \delta\omega^2 = 0,$$

we find that $\mathbf{n}\omega_t^1 = d\mathbf{n}\omega^2$, so likewise,

$$\mathbf{n}\omega^1(x,t) = \mathbf{n}\omega^1(0) + \int_0^t d\mathbf{n}\omega^2(x,t')dt', \quad x \in \partial M. \tag{20}$$

In the sequel we shall mainly consider the case $\omega(0) = 0$, when the lateral Cauchy data for the original problem of electrodynamics is simply

$$\mathbf{t}\omega = (0, f, -\int_0^t df(t')dt'), \tag{21}$$

$$\mathbf{n}\omega = (0, g, \int_0^t dg(t')dt') \tag{22}$$

where f and g are functions of t with values in $\Omega^1\partial M$.

The following theorem implies that solutions of Maxwell's equations are solutions of the complete Maxwell system and gives sufficient conditions for the converse result.

Theorem 1.7 *If $\omega(t) \in C(\mathbb{R}, \mathbf{H}^1) \cap C^1(\mathbb{R}, \mathbf{L}^2)$ satisfies the equation*

$$\omega_t + \mathcal{M}\omega = 0, \quad t > 0 \tag{23}$$

with vanishing initial data $\omega(0) = 0$, and $\omega^0(t) = 0$, $\omega^3(t) = 0$, then the Cauchy data is of the form (21)–(22).

Conversely, if the lateral Cauchy data is of the form (21)–(22) for $0 \leq t \leq T$, and ω satisfies the equation (23), with vanishing initial data, then $\omega(t)$ is a solution to Maxwell's equations, i.e., $\omega^0(t) = 0$, $\omega^3(t) = 0$.

Proof: The first part of the theorem follows from the above considerations if we show that $\omega(t)$ is sufficiently regular.

Since $\omega^2 \in C(\mathbb{R}, H^1(\Omega^2 M))$ we see that $\mathbf{n}\omega^2 \in C(\mathbb{R}, H^{1/2}(\Omega^2 \partial M))$ with $d\mathbf{n}\omega^2 \in C(\mathbb{R}, H^{-1/2}(\Omega^2 \partial M))$. Furthermore, as $\delta\omega_t^1(t) = \delta\delta\omega^2(t) = 0$,

$$\mathbf{n}\omega_t^1 \in C(\mathbb{R}, H^{-1/2}(\Omega^2 \partial M)), \quad \delta\omega^2 \in C(\mathbb{R}, H^{-1/2}(\Omega^2 \partial M)),$$

which verifies (22).

To prove (21) we use Maxwell duality: Consider the forms

$$\eta^{3-k} = (-1)^k * \frac{1}{\alpha}\omega^k.$$

Then $\eta = (\eta^0, \eta^1, \eta^2, \eta^3)$ satisfies Maxwell's equations $\eta_t + \widetilde{\mathcal{M}}\eta = 0$ where $\widetilde{\mathcal{M}}$ is the Maxwell operator with metric g and scalar impedance α^{-1}. In sequel, we call Maxwell's equation with these parameters the adjoint Maxwell equations and the forms η^j the adjoint solution. Now the formula (22) for adjoint solution implies (21) for ω.

To prove the converse, it suffices to show that $\omega^0(t) = 0$. Indeed, the claim $\omega^3(t) = 0$ follows then by Maxwell duality described earlier. From the equations

$$\omega_t^0 - \delta\omega^1 = 0, \tag{24}$$
$$\omega_t^1 + d\omega^0 - \delta\omega^2, = 0 \tag{25}$$

it follows that ω^0 satisfies the wave equation

$$\omega_{tt}^0 + \delta d\omega^0 = 0.$$

It also satisfies the initial condition $\omega^0(0) = 0$ and $\omega_t^0(0) = 0$ and, from (21), boundary condition $\mathbf{t}\omega^0 = 0$. Thus, $\omega^0 = 0$ for $0 \le t \le T$.

□

The following definition fixes the solution of the forward problem considered in this work.

Definition 1.8 *Let $f = (f^0, f^1, f^2) \in C^\infty([0,T]; \mathbf{\Omega}(\partial M))$ be a smooth boundary source of the form (21), i.e., $f^0 = 0$, $f_t^2 = -df^1$. Further, let R be any right inverse of the mapping $\mathbf{t}$. The solution of the initial-boundary value problem*

$$\omega_t + \mathcal{M}\omega = 0, \quad t > 0,$$
$$\omega(0) = \omega_0 \in \mathbf{L}^2(M), \quad \mathbf{t}\omega = f,$$

is given by

$$\omega = Rf + \mathcal{U}(t)\omega_0 - \int_0^t \mathcal{U}(t-s)(\mathcal{M}Rf(s) + Rf_s(s))ds.$$

We remark that the boundary data f could be chosen from a wider class $f \in H^{1/2}(\partial M \times [0,T])$.

Theorem 1.7 motivates the following definition.

Definition 1.9 *For solution ω of Maxwell equations (10)–(11) we use the following notations:*

i. The lateral Cauchy data for a solution ω of Maxwell's equations with vanishing initial data in the interval $0 \le t \le T$ is given by the pair

$$(\mathbf{t}\omega^1(x,t), \mathbf{n}\omega^2(x,t)), \quad (x,t) \in \partial M \times [0,T].$$

ii. When ω satisfies initial condition $\omega(0) = 0$ the mapping

$$\mathcal{Z}^T : C_{00}^\infty([0,T], \Omega^1(\partial M)) \to C_{00}^\infty([0,T], \Omega^1(\partial M)),$$

$$\mathcal{Z}^T(\mathbf{t}\omega^1) = \mathbf{n}\omega^2|_{\partial M \times [0,T]},$$

is well defined. We call this map the admittance map.

Here $C_{00}^\infty([0,T], B)$ consists of C^∞ functions of t with values in a space B, i.e. $B = \Omega^1(\partial M)$ in definition 1.9, which vanish near $t = 0$.

Note that in the classical terminology for the electric and magnetic fields, $\mathcal{Z}^T$ maps the tangential electric field $n \times E|_{\partial M \times [0,T]}$ to the tangential magnetic field $n \times H|_{\partial M \times [0,T]}$.

The boundary data and the energy of the field inside M are closely related. The following result, crucial from the point of view of boundary control, is a version of the Blagovestchenskii formula (see [5] for the case of the scalar wave equation). Observe that the following theorem is formulated for any solutions of the complete system, not only for those that correspond to Maxwell's equations.

Theorem 1.10 *Let ω and η be smooth solutions of the complete system (14). Then the knowledge of the lateral Cauchy data*

$$(\mathbf{t}\omega, \mathbf{n}\omega), \quad (\mathbf{t}\eta, \mathbf{n}\eta), \quad 0 \le t \le 2T,$$

is sufficient for the determination of the inner products

$$(\omega^j(t), \eta^j(s))_{L^2}, \quad 0 \le j \le 3, \ 0 \le s, t \le T$$

over the manifold M.

Proof: The proof is based on the observation that, having the lateral Cauchy data of a solution ω, we also have access to the forms $d\mathbf{t}\omega$ and $d\mathbf{n}\omega$ at the boundary. On the other hand, $\mathbf{t}$ commutes with d so that

$$\mathbf{t}d\omega^j = d\mathbf{t}\omega^j, \quad \mathbf{n}\delta\omega^j = \mathbf{t} * *d\frac{1}{\alpha} * \omega^j = d\mathbf{t}\frac{1}{\alpha} * \omega^j = d\mathbf{n}\omega^j.$$

Let us define the function

$$F^j(s,t) = (\omega^j(s), \eta^j(t)).$$

From the complete system, it follows that

$$\begin{aligned}(\partial_s^2 - \partial_t^2)F^j(s,t) &= (\omega^j_{ss}(s), \eta^j(t))_{L^2} - (\omega^j(s), \eta^j_{tt}(t))_{L^2} \qquad (26)\\ &= -((d\delta + \delta d)\omega^j(s), \eta^j(t))_{L^2} + (\omega^j(s), (d\delta + \delta d)\eta^j(t))_{L^2}\\ &= f^j(s,t).\end{aligned}$$

By applying Stokes theorem we obtain further that

$$f^j(s,t) = \langle \mathbf{n}\omega^j, \mathbf{t}\delta\eta^j\rangle + \langle \mathbf{t}\omega^j, \mathbf{n}d\eta^j\rangle - \langle \mathbf{t}\delta\omega^j, \mathbf{n}\eta^j\rangle - \langle \mathbf{n}d\omega^j, \mathbf{t}\eta^j\rangle,$$

where we suppressed for brevity the dependence of the boundary values on s and t. Now the complete system implies that

$$d\omega^j = -\omega^{j+1}_s + \delta\omega^{j+2}, \quad \delta\omega^j = \omega^{j-1}_s + d\omega^{j-2},$$

and, similarly,

$$d\eta^j = -\eta^{j+1}_t + \delta\eta^{j+2}, \quad \delta\eta^j = \eta^{j-1}_t + d\eta^{j-2}.$$

A substitution to the above formulas then gives

$$\begin{aligned}f^j(s,t) &= \langle \mathbf{n}\omega^j, \mathbf{t}\eta^{j-1}_t + d\mathbf{t}\eta^{j-2}\rangle + \langle \mathbf{t}\omega^j, -\mathbf{n}\eta^{j+1}_t + d\mathbf{n}\eta^{j+2}\rangle\\ &\quad -\langle \mathbf{t}\omega^{j-1}_s + d\mathbf{t}\omega^{j-2}, \mathbf{n}\eta^j\rangle - \langle -\mathbf{n}\omega^{j+1}_s + d\mathbf{n}\omega^{j+2}, \mathbf{t}\eta^j\rangle,\end{aligned}$$

where d stands for the exterior derivative on ∂M. Hence, f^j is completely determined by the lateral Cauchy data. What is more, we have

$$F^j(0,t) = F^j(s,0) = 0, \quad F^j_s(0,t) = F^j_t(s,0) = 0. \qquad (27)$$

Hence, we can solve $F(s,t)$ using (26) and (27) as claimed. □

Remark 3. If ω and η are solutions to Maxwell's equations (10-11), the formulas above simplify. We have

$$f^0(s,t) = f^3(s,t) = 0,$$

and

$$f^1(s,t) = \langle \mathbf{n}\omega^2_s, \mathbf{t}\eta^1\rangle - \langle \mathbf{t}\omega^1, \mathbf{n}\eta^2_t\rangle, \quad f^2(s,t) = \langle \mathbf{n}\omega^2, \mathbf{t}\eta^1_t\rangle - \langle \mathbf{t}\omega^1_s, \mathbf{n}\eta^2\rangle.$$

Then, for $j=1$ the inner product $(\omega^1(t), \omega^1(t))_{L^2}$ defines the energy of the electric field. Similarly, for $j=2$ the inner product $(\omega^2(t), \omega^2(t))_{L^2}$ defines the energy of the magnetic field.

2 Inverse problem

The main objective of this chapter is to prove the following uniqueness result for the inverse boundary value problem.

Theorem 2.1 *Given ∂M and the admittance map $\mathcal{Z}_T$, $T > 8diam\,(M)$, for Maxwell's equations, (10)– (11), it is possible to uniquely reconstruct the Riemannian manifold, (M, g) and the scalar wave impedance, α.*

Observe that, once we know the travel time metric g as well as the wave impedance α, formula (8) gives the metrics g_μ and g_ϵ, which correspond to the material parameters μ and ϵ.

The proof of the above result is divided in several parts. The first step, which is discussed in the next sections, is to prove necessary boundary controllability results. These results are used, in a similar fashion as in [27], [25], to reconstruct the manifold and the travel time metric.

2.1 Unique continuation results

In the following lemma, we consider extensions of differential forms outside the manifold M. Let $\Gamma \subset \partial M$ be open. Assume that $\widetilde{M}$ is an extension of M across Γ, i.e. $M \subset \widetilde{M}$, $\Gamma \subset \text{int}(\widetilde{M})$ and $\partial M \setminus \Gamma \subset \partial \widetilde{M}$. Furthermore, we assume that the metric g and impedance α are extended smoothly into $\widetilde{M}$ as $\widetilde{g}$, $\widetilde{\alpha}$. In this case, we say that the manifold with scalar impedance $(\widetilde{M}, \widetilde{g}, \widetilde{\alpha})$ is an *extension of* (M, g, α) *across* Γ. (See Figure 2.1).

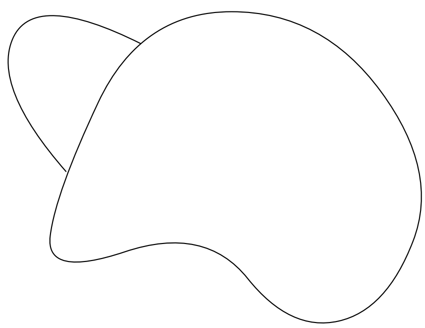

Figure 1: Manifold $\widetilde{M}$ is obtained by gluing an "ear" to M.

We have the following simple result.

Lemma 2.2 *Assume that $\widetilde{M}$ is an extension of M across an open set $\Gamma \subset \partial M$. Let ω^k be a k-form on M and $\widetilde{\omega}^k$ be its extension by zero to $\widetilde{M}$. Then*

1. *If $\omega^k \in H(d, \Omega^k M)$ and $\mathbf{t}\omega^k|_\Gamma = 0$, then $\widetilde{\omega}^k \in H(d, \Omega^k \widetilde{M})$.*

2. If $\omega^k \in H(\delta, \Omega^k M)$ and $\mathbf{n}\omega^k|_\Gamma = 0$, then $\widetilde{\omega}^k \in H(\delta, \Omega^k \widetilde{M})$.

Proof: External differential, in terms of distributions, of $\widetilde{\omega}^k$ can be defined by

$$(d\widetilde{\omega}^k, \varphi^k)_{L^2} = (\widetilde{\omega}^k, \delta\varphi)_{L^2},$$

where $\varphi^k \in \Omega^k \widetilde{M}^{\mathrm{int}}$ is arbitrary. However, by the formula (16),

$$(\widetilde{\omega}^k, \delta\varphi^k)_{L^2(\widetilde{M})} = (\omega^k, \delta\varphi^k)_{L^2(M)} = (d\widetilde{\omega}^k, \varphi^k)_{L^2(M)} + \langle \mathbf{t}\omega^k, \mathbf{n}\varphi^k \rangle.$$

Moreover, since $\mathrm{supp}\,(\varphi^k) \subset\subset \widetilde{M}^{\mathrm{int}}$, then $\mathrm{supp}(\mathbf{n}\varphi^k) \subset \Gamma$, where $\mathbf{t}\omega^k$ vanishes. Thus,

$$(\widetilde{\omega}^k, \delta\varphi^k)_{L^2(\widetilde{M})} = (d\omega^k, \varphi^k)_{L^2},$$

i.e. $d\widetilde{\omega}^k$ is the zero extension of $d\omega^k$. In particular, $d\widetilde{\omega}^k \in L^2(\widetilde{M})$, so that $\widetilde{\omega}^k \in H(d, \Omega^k \widetilde{M})$.

The claim concerning the codifferential is proved by a similar argument. □

As a consequence of this result, we obtain the following.

Theorem 2.3 *Let $\omega \in C^1(\mathbb{R}, \mathbf{L}^2) \cap C(\mathbb{R}, \mathbf{H})$, $\mathbf{t}\omega|_{\Gamma\times[0,T]} = 0$, $\mathbf{n}\omega|_{\Gamma\times[0,T]} = 0$, be a solution of the equation $\omega_t + \mathcal{M}\omega = 0$ in $M \times [0,T]$. Let $\widetilde{\omega}$ be its extension by zero across $\Gamma \subset \partial M$. Then the extended form, $\widetilde{w}(t)$ satisfies the complete Maxwell's system on $(\widetilde{M}, \widetilde{g}, \widetilde{\alpha})$, i.e. $\widetilde{\omega}_t + \widetilde{\mathcal{M}}\widetilde{\omega} = 0$ in $\widetilde{M} \times [0,T]$.*

We are particularly interested in the solutions of Maxwell's equations. The following result is not directly needed but we have included it, since the basic idea is useful when we will prove the main result of this section.

Lemma 2.4 *Assume that ω in the above theorem satisfies Maxwell's equations, i.e., $\omega^0 = 0$ and $\omega^3 = 0$, and $\omega(x,0) = 0$. If $\mathbf{t}\omega^1 = 0$ and $\mathbf{n}\omega^2 = 0$ on $\Gamma \times [0,T]$, then ω satisfies Maxwell's equations in the extended domain $\widetilde{M} \times [0,T]$.*

Proof: From Theorem 1.7 it follows that, since ω satisfies Maxwell's equations,

$$\mathbf{t}\omega = (0, \mathbf{t}\omega^1, -\int_0^t d\mathbf{t}\omega^1 dt') = 0, \quad \mathbf{n}\omega = (0, \mathbf{n}\omega^2, \int_0^t d\mathbf{n}\omega^2 dt') = 0$$

in $\Gamma \times [0,T]$. Therefore, the previous theorem shows that the continuation by zero across Γ, $\widetilde{\omega}(t)$, satisfies the complete system in $\widetilde{M} \times [0,T]$.

However, $\widetilde{\omega}^0(t) = 0$, $\widetilde{\omega}^3(t) = 0$ in $\widetilde{M} \times [0,T]$, i.e., $\widetilde{\omega}(t)$ satisfies Maxwell's equations with vanishing initial data in the extended manifold $\widetilde{M}$. □

When we deal with a general solution to Maxwell's equations, (10)–(11), which may not satisfy zero initial conditions, and try to extend them by zero across Γ, the arguments of Lemma 2.4 fail. Indeed, if $\omega_0 \neq 0$, then (20) show that $\mathbf{n}\omega^2 = 0$ is not sufficient for $\mathbf{n}\omega^1 = 0$. However, by differentiating with respect to time, the parasite term $\mathbf{n}\omega^1(0)$ vanishes. This is the motivation why, in the following theorem, we consider the time derivatives of the weak solutions.

Denote by $\tau(x, y)$ the geodesic distance between x and y on (M, g). Let $\Gamma \subset \partial M$ be open and $T > 0$. We use the notation

$$K(\Gamma, T) = \{(x, t) \in M \times [0, 2T] \mid \tau(x, \Gamma) < T - |T - t|\}$$

for the double cone of influence with base on the slice $t = T$. (see Figure 2.1.)

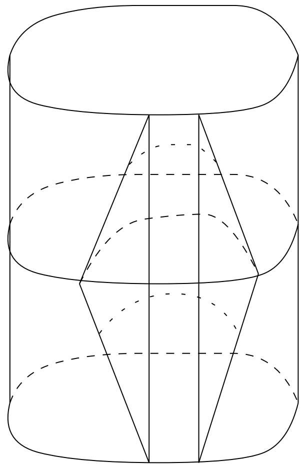

Figure 2: Double cone of influence.

Theorem 2.5 *Let $\omega(t)$ be a weak solution of Maxwell's system in the sense of Definition 1.5 with $\omega_0 = (0, \omega_0^1, \omega_0^2, 0)$. Assume, in addition, that $\delta\omega_0^1 = 0$, $d\omega_0^2 = 0$ and $\rho = 0$. If $\mathbf{n}\omega^2 = 0$ in $\Gamma\times]0, 2T[$, then $\partial_t\omega = 0$ in the double cone $K(\Gamma, T)$.*

Proof: Let $\psi \in C_0^\infty([-1, 1])$, $\int_{-1}^1 \psi(s)ds = 1$ be a Friedrich's mollifier. Then, for any $\sigma > 0$ and $\omega(t) \in C(]0, 2T[), \mathbf{L}^2(M))$ satisfying conditions of the Theorem, denote by $\omega_\sigma(t)$ its time-regularization,

$$\omega_\sigma = \psi_\sigma * \omega, \quad \psi_\sigma(t) = (1/\sigma)\psi(t/\sigma).$$

Then $\omega_\sigma \in C^\infty([\sigma, 2T - \sigma[, \mathbf{L}^2(M))$ continue to be weak solutions to the Maxwell system and, moreover, to Maxwell's equations (10)–(11). Thus,

$$\mathcal{M}\omega_\sigma = -\partial_t\omega_\sigma \in C^\infty([\sigma, 2T - \sigma[, \mathbf{L}^2(M)),$$

i.e. $\omega_\sigma \in C^\infty([\sigma, 2T - \sigma[, \mathcal{D}(\mathcal{M}_\mathrm{e}))$. Repeating these arguments,

$$\omega_\sigma \in C^\infty([\sigma, 2T - \sigma[, \mathcal{D}(\mathcal{M}_\mathrm{e}^\infty)),$$

with $\mathcal{D}(\mathcal{M}_{\rm e}^\infty) = \bigcap_{N>1} \mathcal{D}(\mathcal{M}_{\rm e}^N)$.

As $\mathbf{n}\omega_\sigma = \psi_\sigma * \mathbf{n}\omega$,

$$\mathbf{n}(\omega_\sigma)^2 = 0 \quad \text{on } \Gamma \times [\sigma, 2T-\sigma[.$$

Applying (20), we see that $\mathbf{n}\partial_t \omega_\sigma = 0$ on $\Gamma \times [\sigma, 2T-\sigma[$.

Denote by $\widetilde{\omega}$ the extension by zero of ω across Γ and $\widetilde{\eta}_\sigma$ that of $\partial_t\omega_\sigma$. We claim that, in the distributional sense, $\widetilde{\eta}_\sigma$ satisfies the complete Maxwell system, for $\sigma < t < 2T-\sigma$. Indeed, let $\varphi = (\varphi^0, \varphi^1, \varphi^2, \varphi^3) \in C_0^\infty(]\sigma, 2T-\sigma[, \mathbf{\Omega}\widetilde{M}^{\rm int})$ be a test form. Using the brackets $[\,\cdot\,,\,\cdot\,]$ to denote the distribution duality, that extends the inner product

$$[\psi, \phi] = \int_0^{2T} (\psi(t), \phi(t))_{\mathbf{L}^2(\widetilde{M})} dt,$$

we have

$$\begin{aligned} [\partial_t \widetilde{\eta}_\sigma + \widetilde{\mathcal{M}}\widetilde{\eta}_\sigma, \varphi] &= -[\widetilde{\eta}_\sigma, \widetilde{\mathcal{M}}\varphi + \varphi_t] \\ = [\widetilde{\omega}_\sigma, \widetilde{\mathcal{M}}(\varphi_t) + \varphi_{tt}] &= [\omega_\sigma, \mathcal{M}(\varphi_t) + \varphi_{tt}]. \end{aligned}$$

As $\mathbf{t}\omega_\sigma = 0$, it follows from the Stokes' theorem and the fact that ω_σ satisfies Maxwell's equations, that

$$[\omega, \mathcal{M}\varphi_t + \varphi_{tt}] = \int_0^{2T} (\omega_\sigma, \mathcal{M}\varphi_t + \varphi_{tt})_{\mathbf{L}^2(M)} dt = \int_0^{2T} \langle \mathbf{n}\omega_\sigma . \mathbf{t}\varphi_t \rangle dt,$$

As $\mathrm{supp}(\mathbf{t}\varphi) \subset \Gamma \times]\sigma, 2T-\sigma[$, where $\mathbf{n}\omega_\sigma = 0$, the right side of this equation equals to 0. In addition, $\widetilde{\mathbf{t}}\widetilde{\omega}_\sigma = 0$ for $t \in]\sigma, 2T-\sigma[$, where $\widetilde{\mathbf{t}}$ is the tangential component on $\partial\widetilde{M}$. Thus, the claim follows.

However, $\widetilde{\eta}_\sigma \in C^\infty(]\sigma, 2T-\sigma[, \mathbf{L}^2(\widetilde{M}))$. Therefore, similar considerations to the above shows that this implies that

$$\widetilde{\eta}_\sigma \in C^\infty([\sigma, 2T-\sigma[, \mathcal{D}^\infty(\mathcal{M}_{\rm e})),$$

i.e. $\widetilde{\eta}_\sigma$ is infinitely smooth in $\widetilde{M}^{\rm int} \times [\sigma, 2T-\sigma[$. Since $\widetilde{\eta}_\sigma = 0$ outside $M \times \mathbb{R}$, the unique continuation result of Eller-Isakov-Nakamura-Tataru [17], that is based on result of Tataru [55],[57]

for smooth solutions, implies that $\widetilde{\eta}_\sigma = 0$ in the double cone $\widetilde{\tau}(x, \widetilde{M} \setminus M) < T - \sigma - |T-t|, \quad x \in \widetilde{M}$, where $\widetilde{\tau}$ is the distance on $(\widetilde{M}, \widetilde{g})$. As $\widetilde{\eta}_\sigma = \partial_t\omega_\sigma$ in M, this implies that $\partial_t\omega_\sigma = 0$ in the double cone

$$\tau(x, \Gamma) < T - \sigma - |T-t|, \quad x \in M. \tag{28}$$

When $\sigma \to 0$, $\widetilde{\eta}_\sigma \to \partial_t\omega$, in the distributional sense, while the cone (28) tend to $K(\Gamma, T)$ and the claim of the theorem follows. $\square$

We note that the unique continuation result of [17] is related to scalar ϵ, μ. However, it is easily generalized to the scalar impedance case due to the single velocity of the wave propagation.

Following the proof of Theorem 2.5, we can show the following variant of Theorem 1.7.

Corollary 2.6 *Let $\omega(t)$ be a weak solution to the complete Maxwell system in the sense of definition 1.5, with $\rho = 0$, and, in addition, (22) on $\Gamma \times]0, T[$. If $T > 2\, diam(M)$, then $\omega^0(t) = 0$, $\omega^3(t) = 0$ and $\omega(t)$ is a solution of Maxwell's system for $0 < t < T$.*

Proof: We will consider only ω^0 using the n Maxwell duality for ω^3. By remark 1 and (21),

$$\omega^0_{tt} + \delta d\omega^0 = 0, \quad \mathbf{t}\omega^0|_{\partial M\times[0,T]} = 0. \tag{29}$$

Also

$$\omega^1_t + d\omega^0 - \delta\omega^2 = 0,$$

imply, together with (22), that

$$\mathbf{n}d\omega^0 = \mathbf{n}\delta\omega^2 - \mathbf{n}\omega^1_t = d\mathbf{n}\omega^2 - \mathbf{n}\omega^1_t = 0$$

on $\Gamma \times [0, T]$. Together with the boundary condition in (29), this shows that the lateral Cauchy data of $\omega(t)$ vanishes on $\Gamma \times [0, T]$. Using now the wave equation in (29), this imply that, due to Tataru's unique continuation [55], [57], $\omega_0 = 0$ in the double cone $K(\Gamma, T)$. As $T > 2\mathrm{diam}(M)$, this yield that $\omega^0(T/2) = \omega^0_t(T/2) = 0$. It now follows from (29) that $\omega^0(t) = 0$ for $0 < t < T$. □

2.2 Introduction for controllability

In this section we derive the controllability results for the Maxwell system. We divide these results in *local results*, i.e., controllability of the solutions at short times and in *global results*, where the time of control is long enough so that the controlled electromagnetic waves fill the whole manifold. Both types of results are based on the unique continuation of Theorem 2.5 and representation of inner products of electromagnetic fields over M, in a time slice, in terms of integrals of the lateral Cauchy data over the boundary ∂M over a time interval which is given by Theorem 1.10 .

Consider the initial boundary value problem

$$\omega_t + \mathcal{M}\omega = 0, \quad t > 0, \tag{30}$$

with the initial data $\omega(0) = 0$ and the electric boundary data of Maxwell type,

$$\mathbf{t}\omega = (0, f, -\int_0^t df(t')dt'), \tag{31}$$

where we assume that $f \in C_0^\infty(\mathbb{R}_+, \Omega^1 \partial M)$. By Theorem 1.7, we know that $\omega^0(t) = 0$ and $\omega^3(t) = 0$.

Let $\widetilde{\omega}$ denote the weak solution of Definition 1.5 with $\rho = 0$ and $\widetilde{\omega}(T) = \omega_0$. Assume, in addition, that the conditions of Lemma 1.6 are satisfied so that $\widetilde{\omega}$ satisfies also Maxwell's equations.

As we have seen, Stokes formula implies the identity

$$(\omega(T), \omega_0) = -\int_0^T \langle \mathbf{t}\omega, \mathbf{n}\widetilde{\omega} \rangle dt. \tag{32}$$

We refer to this identity as the *control identity* in the sequel.

2.3 Local controllability

In this section, we study differential 1−forms in M that can be generated by using appropriate boundary sources active for short periods of time. Instead of a complete characterization of these forms, we show that there is a large enough subspace in $L^2(\Omega^1 M)$ that can be produced by boundary sources. The difficulty that prevents a complete characterization is related to the topology of the domain of influence, which can be very complicated.

Let $\Gamma \subset \partial M$ be an open subset of the boundary and $T > 0$ arbitrary. We define the *domain of influence* as

$$M(\Gamma, T) = \{x \in M \mid \tau(x, \Gamma) < T\},$$

where τ is the distance with respect to the travel time metric g. Observe that $M(\Gamma, T) = K(\Gamma, T) \cap \{t = T\}$.

Furthermore, let ω be the strong solution of the initial-boundary value problem

$$\omega_t + \mathcal{M}\omega = 0, \quad \omega(0) = 0,$$

with the boundary value

$$\mathbf{t}\omega = (0, f, -\int_0^t df(t')dt'),$$

where $f \in C_0^\infty(]0, T[, \Omega^1 \Gamma)$ with $C_0^\infty(]0, T[, \Omega^1 \Gamma)$ being a subspace of forms in $C_0^\infty(]0, T[, \Omega^1 \partial M)$ with support in Γ. To emphasize the dependence of ω on f, we write occasionally

$$\omega = \omega^f = (0, (\omega^f)^1, (\omega^f)^2, 0).$$

We denote

$$X(\Gamma, T) = \mathrm{cl}_{L^2}\{(\omega^f)^1(T) \mid f \in C_0^\infty(]0, T[, \Omega^1 \Gamma)\}, \tag{33}$$

i.e., $X(\Gamma,T)$ is the L^2–closure of the set of the electric fields that are generated by C_0^∞–boundary sources on $\Gamma\times]0,T[$. Furthermore, we use the notation

$$H(\delta,M(\Gamma,T))=\{\omega^2\in H(\delta,M),\ \mathrm{supp}\,(\omega^2)\in\overline{M(\Gamma,T)}\}.$$

We will prove the following result.

Theorem 2.7 *The set $X(\Gamma,T)$ satisfies*

$$\delta H_0^1(\Omega^2M(\Gamma,T))\subset X(\Gamma,T)\subset \mathrm{cl}_{L^2}\Big(\delta H(\delta,M(\Gamma,T))\Big).$$

Here $H_0^1(\Omega^2S)$, $S\subset M$ is a subspace of $H_0^1(\Omega^2M)$ of forms with support in $\mathrm{cl}(S)$.

Proof: The right inclusion is straightforward: Since ω satisfies Maxwell's equations, we have $\omega^0(t)=0$ and, hence,

$$\omega^1(T)=\int_0^T\delta\omega^2dt\in\delta H(\delta,M(\Gamma,T)).$$

To prove the left inclusion, we show that any field of the form $\nu^1=\delta\eta^2$ with $\eta^2\in H_0^1(M(\Gamma,T))$ is in $(X(\Gamma,T)^\perp)^\perp$. To this end, let us first assume that $\omega_0^1\in L^2(\Omega^1M)$ is a 1–form such that

$$(\omega_0^1,\omega^1)_{L^2}=0$$

for all $\omega^1=(\omega^f(T))^1$ generated by boundary sources $f\in C_0^\infty(]0,T[,\Omega^1\Gamma)$. Since $\omega^1=\delta\omega^2$, it suffices to consider only those forms ω_0^1 that are of the form $\omega_0^1=\delta\eta_0^2$ for some $\eta_0^2\in H(\delta)$. Indeed, by Hodge decomposition (see [51]) in $\mathbf{L}^2(M)$, we have

$$\omega_0^1=\widehat{\omega}_0^1+\delta\eta_0^2,$$

where $d\widehat{\omega}_0^1=0$, $\mathbf{t}\widehat{\omega}_0^1=0$ so $\widehat{\omega}_0^1\perp\omega^1$ automatically.

Let $\widetilde{\omega}$ be a weak solution, at the time interval $[0,T]$, of the initial boundary value problem (19) with $\mathbf{t}\widetilde{\omega}=0$, and

$$\widetilde{\omega}(\,\cdot\,,T)=(0,\omega_0^1,0,0)=\omega_0.$$

By our assumption,

$$(\omega(T),\omega_0)_{\mathbf{L}^2}=(\omega^1(T),\omega_0^1)_{L^2}=0,$$

and thus, by the control identity, (32) and conditions (21), (22),

$$\int_0^T\langle\mathbf{t}\omega,\mathbf{n}\widetilde{\omega}\rangle=\int_0^T\langle\mathbf{t}\omega^1,\mathbf{n}\widetilde{\omega}^2\rangle=\int_0^T\langle f,\mathbf{n}\widetilde{\omega}^2\rangle=0,$$

for all differential 1-forms $f \in C_0^\infty(]0,T[,\Omega^1\Gamma)$. Thus, we have

$$\mathbf{n}\widetilde{\omega}^2 = 0 \text{ on } \Gamma\times]0,T[.$$

Furthermore, it is easy to see that, for $T+t \in [T,2T]$, we have

$$\widetilde{\omega}(T+t) = (0, \widetilde{\omega}^1(T-t), -\widetilde{\omega}^2(T-t), 0),$$

and, therefore, also

$$\mathbf{n}\widetilde{\omega}^2 = 0 \text{ on } \Gamma\times]T,2T[.$$

But this implies that, as a distribution, $\mathbf{n}\widetilde{\omega}^2$ vanishes on the whole interval $]0,2T[$ since it is in $L^2_{\rm loc}(\mathbb{R}, H^{-1/2}(\partial M))$. By applying the Theorem 2.5, we can deduce that $\widetilde{\omega}_t = 0$ in the double cone $K(\Gamma,T)$. In particular, we have that $d\omega_0^1 = \widetilde{\omega}_t^2(T) = 0$ in $M(\Gamma,T)$.

Let now $\nu^1 = \delta\eta^2 \in \delta H_0^1(\Omega^2 M(\Gamma,T))$. Then

$$(\nu^1, \omega_0^1)_{L^2} = (\eta^2, d\omega_0^1)_{L^2} = 0.$$

This holds for arbitrary $\omega_0^1 \in X(\Gamma,T)^\perp$, i.e., $\nu \in (X(\Gamma,T)^\perp)^\perp = X(\Gamma,T)$. □

Remark 4. Later in this work, we are mainly interested in controlling the time derivatives of electromagnetic fields. Let us denote

$$\overset{\circ}{C}{}^\infty(\Gamma,T) = \{\int_0^t f(t')dt' \mid f \in C_0^\infty(]0,T[,\Omega^1\Gamma)\}.$$

With this notation, we have

$$X(\Gamma,T) = \mathrm{cl}_{L^2}\{(\omega_t^f(T))^1 \mid f \in \overset{\circ}{C}{}^\infty(\Gamma,T)\}.$$

Indeed, if $\omega^1 = (\omega^f)^1 \in X(\Gamma,T)$, then $(\omega^f)^1 = (\omega_t^F)^1$, where

$$F(t) = \int_0^t f(t')dt'.$$

Conversely, the time derivative of a field ω^f, $f \in \overset{\circ}{C}{}^\infty(\Gamma,T)$ satisfies the initial-boundary value problem with the boundary source $f_t \in C_0^\infty(]0,T[,\Omega^1\Gamma)$.

2.4 Global controllability

We start by introducing some notations. Let ω be the strong solution of the initial-boundary value problem

$$\omega_t + \mathcal{M}\omega = 0, \quad \omega(0) = 0,$$

with the boundary value

$$\mathbf{t}\omega = (0, f, -\int_0^t df(t')dt'),$$

where $f \in C_0^\infty(]0,T_0[,\Omega^1\Gamma)$, $T_0 > 0$ and $\Gamma \subset \partial M$ is an open subset. For $T \geq T_0$, we define

$$Y(\Gamma,T) = \{\omega_t^f(T) \mid f \in C_0^\infty(]0,T_0[,\Omega^1\Gamma)\}. \tag{34}$$

For $\Gamma = \partial M$ we denote $Y(T) = Y(\partial M,T)$. Our objective is to give a characterization of the set $Y(T)$ for T_0 large enough. In the following,

$$\mathrm{rad}(M) = \max_{x\in M} \tau(x,\partial M). \tag{35}$$

We prove the following result.

Theorem 2.8 *Assume that $T_0 > 2\mathrm{rad}(M)$. Then, for $T \geq T_0$, $\mathrm{cl}_{\mathbf{L}^2(M)}Y(T)$ is independent of T, i.e. $\mathrm{cl}_{\mathbf{L}^2(M)}Y(T) = Y$, and, moreover,*

$$Y = \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}. \tag{36}$$

Remark 5. The result holds also for Y replaced with $Y(\Gamma,T)$, when

$$T_0 > 2\max_{x\in M} \tau(x,\Gamma).$$

Proof: Let $\omega = \omega^f$ be a solution. As $f = 0$ for $T \geq T_0$, we have

$$\mathbf{t}\omega^1(T) = 0,$$

and, consequently,

$$\begin{aligned} \omega_t(T) &= -\mathcal{M}\omega(T) = (0, \delta\omega^2(T), -d\omega^1(T), 0) \\ &\in \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}. \end{aligned}$$

To prove the converse inclusion, we show that the space $Y(T)$ is dense in $\{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}$. To this end, let $\omega_0 \in \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}$ and $\omega_0 \perp Y(T)$. This means that, for arbitrary $\omega = \omega^f$ satisfying the initial-boundary value problem (14),

$$(\omega_0, \omega_t(T))_{\mathbf{L}^2} = (\omega_0^1, \omega_t^1(T))_{L^2} + (\omega_0^2, \omega_t^2(T))_{L^2} = 0. \tag{37}$$

Let $\widetilde{\omega}$ denote the weak solution of the problem

$$\begin{aligned} &\widetilde{\omega}_t + \mathcal{M}\widetilde{\omega} = 0, \\ &\mathbf{t}\widetilde{\omega} = 0, \quad \widetilde{\omega}(T) = \omega_0. \end{aligned}$$

Observe that the initial value ω_0 satisfies

$$\delta\omega_0^1 = 0, \quad d\omega_0^2 = 0,$$

which implies that $\widetilde{\omega}$ satisfies Maxwell's equations. Consider the function $F : \mathbb{R} \to \mathbb{R}$,

$$F(t) = (\widetilde{\omega}(t), \omega_t(t))_{\mathbf{L}^2}.$$

We have, by using Maxwell's equations, that

$$\begin{aligned} F_t(t) &= (\widetilde{\omega}, \omega_{tt})_{\mathbf{L}^2} + (\widetilde{\omega}_t, \omega_t)_{\mathbf{L}^2} \\ &= -(\widetilde{\omega}^1, \delta d\omega^1)_{L^2} - (\widetilde{\omega}^2, d\delta\omega^2)_{L^2} + (d\widetilde{\omega}^1, d\omega^1)_{L^2} + (\delta\widetilde{\omega}^2, \delta\omega^2)_{L^2}, \end{aligned}$$

and further, by using Stokes' theorem,

$$F_t(t) = -\langle \mathbf{t}\widetilde{\omega}^1(t), \mathbf{n}d\omega^1(t)\rangle - \langle \mathbf{n}\widetilde{\omega}^2(t), \mathbf{t}\delta\omega^2(t)\rangle.$$

However, $\mathbf{t}\widetilde{\omega} = 0$ and $\delta\omega^2 = \omega^1_t$. Thus,

$$F_t(t) = -\langle \mathbf{n}\widetilde{\omega}^2, \mathbf{t}\omega^1_t\rangle = -\langle \mathbf{n}\widetilde{\omega}^2, f_t\rangle.$$

On the other hand, the initial condition $\omega(0) = 0$, together with the orthogonality condition (37), imply that $F(0) = F(T) = 0$, so that

$$\int_0^T \langle \mathbf{n}\widetilde{\omega}^2, f_t\rangle dt = -\int_0^T F_t(t)dt = 0.$$

Since $f \in C_0^\infty(]0, T[, \Omega^1\Gamma)$ is arbitrary, this implies that

$$\mathbf{n}\widetilde{\omega}^2_t = 0 \text{ in } \Gamma\times]0, T[.$$

But now Theorem 2.5 implies that $\widetilde{\omega}_{tt}$ vanishes in the double cone $K(\Gamma, T/2)$. By the assumption $T_0 > 2\mathrm{rad}(M)$, this double cone contains a cylinder

$$C = M\times]T/2 - s, T/2 + s[$$

with some $s > 0$ (See Figure 2.4). Therefore, $\widetilde{\omega}_{tt}$ that satisfies Maxwell's equations and homogeneous boundary condition $\mathbf{t}\widetilde{\omega}_{tt} = 0$. Therefore, it vanishes in the whole $M \times \mathbb{R}$. In particular, this means that, with some time-independent forms ω_1 and ω_2,

$$\widetilde{\omega}(t) = \omega_1 + t\omega_2,$$

with $\mathbf{t}\omega_1 = 0$, $\mathbf{t}\omega_2 = 0$. Again, by Maxwell's equations, we have

$$\omega_2 = \omega_t = \mathcal{M}\omega_1 + t\mathcal{M}\omega_2,$$

for all t. Therefore,

$$\omega_2 = \mathcal{M}\omega_1, \quad \mathcal{M}\omega_2 = 0.$$

But then, Stokes' theorem implies that

$$(\omega_2, \omega_2)_{\mathbf{L}^2} = (\omega_2, \mathcal{M}\omega_1)_{\mathbf{L}^2} = -(\mathcal{M}\omega_2, \omega_1)_{\mathbf{L}^2} = 0,$$

i.e., $\omega_2 = 0$ and $\mathcal{M}\omega_1 = 0$. Observe that, by the assumption of the Theorem,

$$\omega_1 = \widetilde{\omega}(T) = \omega_0 = (0, -\delta\nu^2, d\nu^1, 0) = \mathcal{M}\nu,$$

for some $\nu \in \{0\} \times \overset{\circ}{H}(d) \times H(\delta) \times \{0\}$. Therefore, a further application of Stokes theorem gives

$$(\omega_1, \omega_1)_{\mathbf{L}^2} = (\omega_1, \mathcal{M}\nu)_{\mathbf{L}^2} = -(\mathcal{M}\omega_1, \nu)_{\mathbf{L}^2} = 0,$$

i.e., also $\omega_1 = \omega_0 = 0$. The proof is therefore complete. □

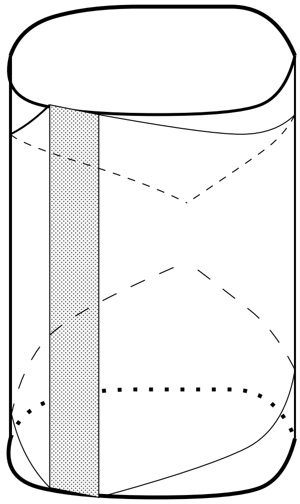

Figure 3: The double cone contains a slice $\{T/2\} \times M$ and the waves vanish near the slice $t = T/2$.

2.5 Generalized sources

So far, we have treated only smooth boundary sources and the corresponding fields. For later use, we need more general boundary sources.

Let $Y = \mathrm{cl}_{\mathbf{L}^2(M)} Y(\partial M, T)$ be the space of the time derivatives of electromagnetic fields satisfying Maxwell's equations, see (34), (36). We define the wave operator

$$W^T : C_0^\infty(]0, T[, \Omega^1 \partial M) \to Y, \quad f \mapsto \omega_t^f(T),$$

where $T \geq T_0$ and $T_0 > 2\,\mathrm{rad}\,(M)$. By means of the wave operator, we define the $\mathcal{F}-$norm on the space of boundary sources as

$$\|f\|_{\mathcal{F}} = \|W^T f\|_{\mathbf{L}^2}. \tag{38}$$

The definition of this norm is independent of the choice of $T \geq T_0$ by conservation of energy.

Notice that by Theorem 1.10, the knowledge of the admittance map $\mathcal{Z}^{2T}$ enables us to calculate explicitly the $\mathcal{F}$–norm of any smooth boundary source.

To complete the space of boundary sources, let us define the equivalence $\sim$ of sources by setting

$$f \sim g \text{ iff } W^T f = W^T g.$$

Further, we define the space $\mathcal{F}([0, T_0])$ as

$$\mathcal{F}([0, T_0]) = C_0^\infty(]0, T[, \Omega^1 \partial M)/\sim .$$

Finally, we complete $\mathcal{F}([0, T_0])$ with respect to the norm (38). Hence, this space, denoted by $\overline{\mathcal{F}}([0, T_0])$ consists of Cauchy sequences with respect to the norm (38), denoted as

$$\widehat{f} = (f_j)_{j=0}^{\infty}, \quad f_j \in C_0^\infty(]0,T[,\Omega^1\partial M).$$

Note that, for any $\widehat{f} \in \overline{\mathcal{F}}$, we can find $\widehat{h} \in \overline{\mathcal{F}}$ such that $\widehat{h} = \widehat{f}$ and $\widehat{h} = (h_j)_{j=1}^{\infty}$, $h_j \in C_0^\infty(]\varepsilon,T[,\Omega^1\partial M)$ for some $\varepsilon > 0$. The reason for this is that Theorem 2.8 is valid also with T_0 replaced with $T_0 - \varepsilon$, when ε is small enough. Thus, for small $\varepsilon > 0$, we can define, for any $\widehat{f} = (f_j)_{j=0}^{\infty} \in \overline{\mathcal{F}}$, the translation $\widehat{f}(\cdot + \varepsilon) = (f_j(\cdot + \varepsilon))_{j=0}^{\infty} \in \overline{\mathcal{F}}$.

These sources are called *generalized sources* in the sequel. The corresponding electromagnetic waves are denoted as

$$\omega_t^{\widehat{f}}(t) = \lim_{j\to\infty} \omega_t^{f_j}(t) \quad \text{for } t \geq T_0. \tag{39}$$

By the isometry of the wave operator, the above limit exists in $\mathbf{L}^2$ for all generalized sources.

We note that the above construction of the space of generalized sources in well-known in PDE-control, e.g. [49], [35].

Remark 6. Observe that since the wave operator W^T is an isometry and $\overline{\mathcal{F}}([0,T_0])$ was defined by closing $C_0^\infty(]0,T[,\Omega^1\partial M)$ with respect to the norm (38), the wave operator extends to a one-to-one isometry

$$\widehat{f} \mapsto \omega_t^{\widehat{f}}(T), \quad \overline{\mathcal{F}}([0,T_0]) \to \mathrm{cl}_{\mathbf{L}^2}(Y(\partial M,T)),$$

where the target space is completely characterized in the previous section.

We say that $\widehat{h} \in \overline{\mathcal{F}}$ is a generalized time derivative of $\widehat{f} \in \overline{\mathcal{F}}$, if for $T = T_0$,

$$\begin{aligned} &\lim_{\sigma\to 0+} \Big|\Big|\frac{\widehat{f}(\cdot+\sigma) - \widehat{f}(\cdot)}{\sigma} - \widehat{h}\Big|\Big|_{\overline{\mathcal{F}}} = \\ &= \lim_{\sigma\to 0+} \Big|\Big|\frac{\omega_t^{\widehat{f}}(T+\sigma) - \omega_t^{\widehat{f}}(T)}{\sigma} - \omega_t^{\widehat{h}}(T)\Big|\Big|_{\mathbf{L}^2(M)} = 0 \end{aligned} \tag{40}$$

In this case we denote $\widehat{h} = \mathbb{D}\widehat{f}$, or just $\widehat{h} = \partial_t\widehat{f}$. In the following, we use spaces $\mathcal{F}^s = \mathcal{D}(\mathbb{D}^s)$, $s \in \mathbb{Z}_+$, which are spaces of generalized sources that have s generalized derivatives. Note that, if (40) is valid for $T = T_0$, it is valid for all $T \geq T_0$ due to the conservation of L^2-norm for Maxwell's equation (energy conservation). Thus, if $\widehat{f} \in \mathcal{F}^s$, we have, for $T \geq T_0$,

$$\mathcal{M}^s\omega_t^{\widehat{f}}(T) = \partial_t^s\omega_t^{\widehat{f}}(t)|_{t=T} \in \mathbf{L}^2(M). \tag{41}$$

Note that $\mathcal{M}$ here is the differential expression given by (15), rather than an operator with some boundary conditions. Since $\mathbf{t}\omega_t^{\widehat{f}} = 0$ on $\partial M \times]T_0,\infty[$, we see that $\mathbf{t}(\partial_t^j\omega_t^{\widehat{f}})(t) = 0$ for $t > T_0$ and $j \leq s-1$. Thus, for $\widehat{f} \in \mathcal{F}^s$ and $T \geq T_0$,

$$\omega_t^{\widehat{f}} \in \bigcap_{j=0}^{s}(C^{s-j}([T,\infty[,\mathcal{D}(\mathcal{M}_e^j)) \cap \mathrm{Ran}\,(\mathcal{M}_e)). \tag{42}$$

Moreover, by (42) and Lemma 1.4,

$$\omega_t^{\widehat{f}}(T) \in \mathbf{H}^s_{loc}(M^{int}) \quad \text{for } T \geq T_0.$$

Next we consider dual spaces to the domains of powers of $\mathcal{M}_e$. Since $\mathbf{H}^s_0 \subset \mathcal{D}(\mathcal{M}^s_{\rm e})$, we have $(\mathcal{D}(\mathcal{M}^s_{\rm e}))' \subset \mathbf{H}^{-s}$. Similarly, we see that $\mathbf{H}^{-s}_0 \subset (\mathcal{D}(\mathcal{M}^s_{\rm e}))'$. These facts will be needed later in the construction of focusing sources.

2.6 Reconstruction of the manifold

In this section we will show how to determine the manifold, M and the travel time metric, g from the boundary measurements of the admittance map $\mathcal{Z}$. We will show that the boundary data determines the set of *boundary distance functions*. The basic idea is to use a slicing principle, when we control the supports of the waves generated by boundary sources.

We start by fixing certain notations. Let times $T_0 < T_1 < T_2$ satisfy

$$T_0 > 2\,\mathrm{rad}(M), \quad T_1 \geq T_0 + \mathrm{diam}(M), \quad T_2 \geq 2\,T_1$$

We assume in this section that the admittance map $\mathcal{Z}^{T_2}$ is known.

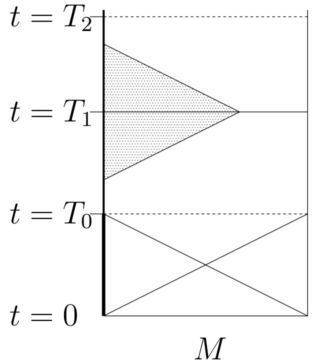

Figure 4: The sources $\widehat{f}$ of the waves $\omega^{\widehat{f}}(x,t)$ are supported on the time-interval $[0, T_0]$ which enables us to control the waves at times $t > T_0$. In the construction of the manifold, the supports of the waves are considered at time $t = T_1$. To this end, we use the unique continuation in double-cones (triangle in the figure) intersecting the boundary in the layer $\partial M \times [T_0, T_1]$. Note that in this layer it is crucial that $\widehat{f} = 0$.

Let $\Gamma_j \subset \partial M$ be open disjoint sets, $1 \leq j \leq J$ and τ_j^- and τ_j^+ be positive times with

$$0 < \tau_j^- < \tau_j^+ \leq \mathrm{diam}(M), \quad 1 \leq j \leq J.$$

We define the set $S = S(\{\Gamma_j, \tau_j^-, \tau_j^+\}) \subset M$ as an intersection of slices,

$$S = \bigcap_{j=1}^{J} \left(M(\Gamma_j, \tau_j^+) \setminus M(\Gamma_j, \tau_j^-) \right). \tag{43}$$

Our first goal is to find out, by boundary measurements, whether the set S contains an open ball or not. To this end, we give the following definition.

Definition 2.9 *The set $Z = Z(\{\Gamma_j, \tau_j^-, \tau_j^+\})_{j=1}^J$ consists of those generalized sources $\widehat{f} \in \mathcal{F}^\infty([0, T_0])$ that produce waves $\omega_t = \omega_t^{\widehat{f}}$ with*

1. $\omega_t^1(T_1) \in X(\Gamma_j, \tau_j^+)$, *for all* j, $1 \le j \le J$,
2. $\omega_t^2(T_1) = 0$,
3. $\omega_{tt}(T_1) = 0$ *in* $M(\Gamma_j, \tau_j^-)$, *for all* j, $1 \le j \le J$.

Remark 7. Observe that, since ω_t satisfies Maxwell's equations, we have, in particular,

$$\omega_{tt}^2 = -d\omega_t^1, \quad \omega_{tt}^1 = \delta\omega_t^2.$$

These identities imply that, at $t = T_1$, ω_{tt} is of the form

$$\omega_{tt}(T_1) = (0, 0, \omega_{tt}^2(T_1), 0) = (0, 0, d\eta^1, 0),$$

for the 1–form $\eta^1 = -\omega_t^1$, and

$$\mathrm{supp}(d\eta^1) \subset S.$$

This observation is crucial later when we will discuss focusing waves.

The central tool for reconstruction the manifold is the following theorem.

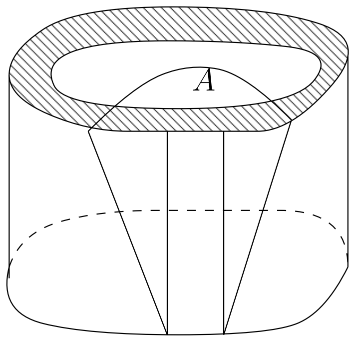

Figure 5: In Definition 2.9 we can consider e.g. the case $\Gamma_1 = \Gamma$, $\tau_1^+ = s_1$, $\tau_1^- = 0$, and $\Gamma_2 = \partial M$, $\tau_2^+ = \mathrm{diam}\,(M)$, $\tau_1^- = s_2$. Then the waves that satisfy the definition have the following properties: By 1., the wave $(\omega_t^{\widehat{f}})^1(T_1)$ coincides with a wave that is supported in $M(\Gamma, s_1)$. This domain of influence on the figure is the upper part of the cone of influence. Thus, $d(\omega_t^{\widehat{f}})^1(T_1) = (\omega_{tt}^{\widehat{f}})^2(T_1)$ is supported in $M(\Gamma, s_1)$. By 2., the wave $(\omega_{tt}^{\widehat{f}})^2(T_1)$ vanish in the boundary layer $M(\partial M, s_2)$. Combining these, we see that $\omega_{tt}^{\widehat{f}}(T_1)$ is supported in $A = M(\Gamma, s_1) \setminus M(\partial M, s_2)$.

Theorem 2.10 *Let S and Z be defined as above. The following alternative holds:*

1. *If S contains an open ball, then* $\dim(Z) = \infty$,
2. *If S does not hold an open ball, then $Z = \{0\}$.*

In order to prove the above alternative, we need the following observability result that will be also useful later.

Theorem 2.11 *Given the boundary map $\mathcal{Z}^{T_2}$, we can determine whether a given boundary source $\widehat{f} \in \mathcal{F}^\infty([0, T_0])$ is in the set Z or not.*

Proof: Let $\widehat{f} = (f_k)_{k=0}^\infty \in \mathcal{F}^\infty([0, T_0])$ be a generalized source. Consider first the question whether $(\omega_t^{\widehat{f}})^1(T_1) \in X(\Gamma_j, \tau_j^+)$. By Remark 4, this is equivalent to the existence of a sequence,

$$\widehat{h} = (h_\ell)_{\ell=0}^\infty, \quad h_\ell \in \overset{\circ}{C}{}^\infty(\Gamma_j, \tau_j^+),$$

such that

$$\lim_{k,\ell \to \infty} \|(\omega_t^{f_k})^1(T_1) - (\omega_t^{h_\ell})^1(\tau_j^+)\| = 0. \tag{44}$$

By linearity of the initial-boundary value problem, we have

$$\|(\omega_t^{f_k})^1(T_1) - (\omega_t^{h_\ell})^1(\tau_j^+)\| = \|(\omega^{g_{k,\ell}})^1(T_1)\|,$$

where the source $g_{k,\ell}$ is

$$g_{k,\ell}(t) = (f_k)_t(t) - (h_\ell)_t(t + \tau_j^+ - T_1) \in C_0^\infty(]0, T_1[; \Omega^1 \partial M).$$

However, by Lemma 1.10, $\|(\omega^{g_{k,\ell}})^1(T_1)\|$ is completely determined by the admittance map, $\mathcal{Z}^{T_2}$ making possible to verify (44).

In a similar fashion, Condition 2 of the definition of Z is valid for $\widehat{f}$, if

$$\lim_{k \to \infty} \|(\omega_t^{f_k}(T_1))^2\| = 0,$$

and this condition can also be verified via the admittance map, $\mathcal{Z}^{T_2}$.

Finally, consider Condition 3. We assume here that we already know that $\widehat{f}$ satisfies Conditions 1–2. First, we observe that $\omega_{tt} = \omega_{tt}^{\widehat{f}}$ satisfies

$$(\partial_t + \mathcal{M})\omega_{tt} = 0 \text{ in } M \times \mathbb{R}_+,$$

along with the boundary condition

$$\mathbf{t}\omega_{tt} = 0 \text{ in } \partial M \times [T_0, \infty[.$$

If Condition 3 holds, by the finite propagation speed, ω_{tt} vanishes in a double cone around Γ_j, i.e.,

$$\omega_{tt} = 0 \text{ in } K_j = \{(x,t) \in M \times \mathbb{R}_+ \mid \tau(x,\Gamma_j) + |t - T_1| < \tau_j^-\},$$

for all $j = 1, \ldots, J$. In particular, this means that, in each K_j, ω_t does not depend on time, and Condition 2 implies that $\omega_t^2 = 0$ in K_j. Hence, we have

$$\mathbf{n}\omega_t^2 = \mathcal{Z}^{T_2} f = 0 \text{ on } \Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[. \tag{45}$$

Conversely, assume that condition (45) holds together with Conditions 1–2. Then ω_t satisfies

$$(\partial_t + \mathcal{M})\omega_t = 0 \text{ in } M \times \mathbb{R}_+$$

with the boundary conditions

$$\mathbf{t}\omega_t^1 = 0, \quad \mathbf{n}\omega_t^2 = 0 \text{ in } \Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[.$$

Here we used the fact that $T_1 - \tau_j^- > T_0$, so that $\widehat{f} = 0$ in $\Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[$. Now the Unique Continuation Principle, given by Theorem 2.5, implies that $\omega_{tt} = 0$ in K_j and, in particular, Condition 3 is valid. The proof is complete as it is clear that the condition (45) is readily observable if the admittance map, $\mathcal{Z}^{T_2}$, $T_2 > T_1 + \tau_j$, is known. □

Now we can give the proof of Theorem 2.10.

Proof of Theorem 2.10: Assume that there is an open ball $B \subset S$. Let $0 \neq \varphi \in \Omega^2 B$ be an arbitrary smooth 2–form with $\text{supp}\,(\varphi) \subset B$. From the global controllability result, Theorem 2.8, it follows the existence of a generalized source $\widehat{f} \in \overline{\mathcal{F}([0, T_0])}$ such that

$$\omega_t^{\widehat{f}}(T_1) = (0, \delta\varphi, 0, 0). \tag{46}$$

Moreover, $\varphi \in \Omega^2 B$ implies that $\varphi \in \mathcal{D}(\mathcal{M}_e^s)$ for any $s > 0$ so that $\widehat{f} \in \mathcal{F}^\infty([0, T_0])$.

We will now show that $\widehat{f} \in Z$. Indeed, Conditions 1–2 are obvious from the definition (46) of $\widehat{f}$. Finally, we observe that

$$\omega_{tt}^{\widehat{f}}(T_1) = -\mathcal{M}\omega_t^{\widehat{f}}(T_1) = (0, 0, -d\delta\varphi, 0),$$

so Condition 3 is also satisfied. This proves the first statement of the theorem.

To prove the second part, assume that S does not contain an open ball. Suppose, on the contrary to the claim, that there is a non-vanishing source $\widehat{f} \in Z$ which produces the wave $\omega(t) = \omega^{\widehat{f}}(t)$. Then, by Conditions 1 and 2 in Definition 2.9,

$$\text{supp}(\omega_t(T_1)) \subset \bigcap_{j=1}^{J} M(\Gamma_j, \tau_j^+) = S^+.$$

Furthermore,

$$\omega_{tt}(T_1) = -\mathcal{M}\omega_t(T_1),$$

so that

$$\mathrm{supp}(\omega_{tt}(T_1)) \subset S^+.$$

On the other hand, Condition 3 in Definition 2.9 imply that

$$\omega_{tt}(T_1) = 0 \text{ in } \bigcup_{j=1}^J M(\Gamma_j, \tau_j^-) = S^-.$$

Thus $\mathrm{supp}\omega_{tt}(T_1) \subset S^+ \setminus S^-$. However, if the set S does not contain an open ball, then the set $S^+ \setminus S^-$ in nowhere dense. Since $\omega_{tt}(T_1)$ is smooth in M^{int}, it vanishes in M. In particular, Maxwell's equations imply that

$$d\omega_t^1(T_1) = -\omega_{tt}^2(T_1) = 0. \tag{47}$$

On the other hand, $\omega_t \in \mathrm{cl}_{\mathbf{L}^2}(Y(\partial M, T_1))$, so Theorem 2.8 implies that $\omega_t^1(T_1)$ is of the form

$$\omega_t^1(T_1) = \delta\eta^2,$$

for some 2–form η. Since, in addition, $\mathbf{t}\omega_t^1(T_1) = 0$ it then follows, by Stokes formula, that

$$(\omega_t^1(T_1), \omega_t^1(T_1))_{L^2} = (\delta\eta^2, \omega_t^1(T_1))_{L^2} = (\eta^2, d\omega_t^1(T_1))_{L^2} = 0.$$

Together with Condition 2, this implies

$$\omega_t(T_1) = 0,$$

contradicting to the assumption $\widehat{f} \neq 0$. The proof is complete. □

We are now ready to construct the set of the boundary distance functions. For each $x \in M$, the corresponding boundary distance function, r_x is a continuous function on ∂M given by

$$r_x : \partial M \to \mathbb{R}_+, \quad r_x(z) = \tau(x, z), \quad z \in \partial M.$$

They define *the boundary distance map* $\mathcal{R} : M \to C(\partial M)$, $\mathcal{R}(x) = r_x$, which is continuous and injective (see [30], [25]). We shall denote the set of all boundary distance functions, i.e., the image of $\mathcal{R}$, by

$$\mathcal{R}(M) = \{r_x \in C(\partial M) \mid x \in M\}.$$

It can be shown (see [30], [25]) that, given $\mathcal{R}(M) \subset L^\infty(\partial M)$ we can endow it, in a natural way, with a differentiable structure and a metric tensor $\widetilde{g}$, so that $(\mathcal{R}(M), \widetilde{g})$ becomes an isometric copy of (M, g),

$$(\mathcal{R}(M), \widetilde{g}) \cong (M, g).$$

Hence, in order to reconstruct the manifold (or more precisely, the isometry type of the manifold), it suffices to determine the set, $\mathcal{R}(M)$, of the boundary distance functions. The following result is therefore crucial.

Theorem 2.12 *Let the admittance map $\mathcal{Z}^{T_2}$ be given. Then, for any $h \in C(\partial M)$, we can find out whether $h \in \mathcal{R}(M)$.*

Proof: The proof is based on a discrete approximation process. First, we observe that the condition $h \in \mathcal{R}(M)$ is equivalent to the condition that for any sampling $z_1, \dots, z_J \in \partial M$ of the boundary, there must be $x \in M$ such that

$$h(z_j) = \tau(x, z_j), \quad 1 \le j \le J.$$

Let us denote $\tau_j = h(z_j)$. By the continuity of the distance function, $\tau(x,z)$ in $x \in M$, $z \in \partial M$, we deduce that the above condition is equivalent to the following one:

For any $\varepsilon > 0$, the points z_j have neighborhoods $\Gamma_j \subset \partial M$ with $\mathrm{diam}(\Gamma_j) < \varepsilon$, such that

$$\mathrm{int}\Big(\bigcap_{j=1}^{J} M(\Gamma_j, \tau_j + \varepsilon) \setminus M(\Gamma_j, \tau_j - \varepsilon) \Big) \neq \emptyset. \tag{48}$$

On the other hand, by Theorem 2.10, condition (48) is equivalent to

$$\dim\Big(Z(\{\Gamma_j, \tau_j + \varepsilon, \tau_j - \varepsilon\} \Big) = \infty,$$

that, by means of Theorem 2.11, can be verified via boundary data. □

As a consequence, we obtain the main result of this section.

Corollary 2.13 *The knowledge of the admittance $\mathcal{Z}^{T_2}$ is sufficient for the reconstruction of the manifold, M endowed with the travel time metric, g.*

Having the manifold reconstructed, the rest of this article is devoted to the reconstruction of the wave impedance, α.

2.7 Focusing sources

In the previous section it was shown that, using boundary data, one can control supports of the 2–forms $(\omega_{tt}^{\widehat{f}})^2(t)$. In this section, the goal is to construct a sequence of sources, $(\widehat{f}_p)$, $p = 1, 2, \cdots$ such that, when $p \to \infty$, the corresponding forms $(\omega_{tt}^{\widehat{f}_p})^2(T_1)$ become supported at a single point, while $(\omega_{tt}^{\widehat{f}_p})^1(T_1) = 0$. For $t \ge T_1$, these fields behave like point sources, a fact that turns out to be useful for reconstructing the wave impedance.

In the following, let $\underline{\delta}_y$ denote the Dirac delta at $y \in M^{\mathrm{int}}$, i.e.,

$$\int_M \underline{\delta}_y(x) \phi(x) dV_g(x) = \phi(y),$$

where $\phi \in C_0^\infty(M)$ and dV_g is volume form of (M, g).

Since the Riemannian manifold (M, g) is already found, we can choose $\Gamma_{jp} \subset \partial M,\ 0 < \tau_{jp}^- < \tau_{jp}^+ < \mathrm{diam}(M)$, so that

$$S_{p+1} \subset S_p, \quad \bigcap_{p=1}^{\infty} S_p = \{y\},\ y \in M^{int}. \tag{49}$$

Then, $Z_p = Z(\{\Gamma_{jp}, \tau_{jp}^-, \tau_{jp}^+\}_{j=1}^{J(p)})$ is the corresponding set of generalized sources defined in Definition 2.9.

Definition 2.14 *Let S_p, $p = 1, 2, \cdots$, satisfy (49). We call the sequence $\widetilde{f} = (\widehat{f}_p)$, $p = 1, 2, \cdots$ with $\widehat{f}_p \in Z_p$, a* focusing sequence *of generalized sources of order s (for brevity, focusing sources), $s \in \mathbb{Z}_+$, if there is a distribution-form $A = A_y$ on M such that*

$$\lim_{p \to \infty} (\omega_t^{\partial_t \widehat{f}_p}(T_1), \eta)_{\mathbf{L}^2} = (A_y, \eta)_{\mathbf{L}^2},$$

for all $\eta \in \mathcal{D}(\mathcal{M}_{\mathrm{e}}^s)$.

Remark 8. Observe that, by the identity,

$$\omega_t^{\partial_t \widehat{f}_p} = \omega_{tt}^{\widehat{f}_p} \tag{50}$$

and Remark 7, the electromagnetic wave $\omega_t^{\partial_t \widehat{f}_p}(T_1)$ is supported in $\mathrm{cl}(S_p)$, so A_y must be supported on $\{y\}$.

We will show the following result.

Lemma 2.15 *Let the admittance map $\mathcal{Z}^{T_2}$ be given. Then, for any $s \in \mathbb{Z}_+$ and any sequence of generalized sources, $(\widehat{f}_p)$, $p = 1, 2 \cdots$, one can determine if $(\widehat{f}_p)$ is a focusing sequence or not.*

Proof: Let $\eta \in \mathcal{D}(\mathcal{M}_{\mathrm{e}}^s)$. We decompose η as $\eta = \eta_1 + \eta_2$, where

$$\eta_1 \in \mathcal{D}(\mathcal{M}_{\mathrm{e}}^s) \cap \mathrm{cl}(Y), \quad \eta_2 \in \mathcal{D}(\mathcal{M}_{\mathrm{e}}^s) \cap Y^{\perp}.$$

By the global controllability result, Theorem 2.8, and isometry of the wave operator W^T, $T \geq T_0$,

$$\eta_1 = \omega_t^{\widehat{h}}, \quad \widehat{h} \in \mathcal{F}^s([0, T_0]).$$

Since $\omega_t^{\partial_t \widehat{f}_p} \in \mathrm{cl}_{\mathbf{L}^2}(Y)$, so that $\omega_t^{\partial_t \widehat{f}_p} \perp \eta_2$, the condition that $\widetilde{f}$ is a focusing source is tantamount to the existence of the limit

$$(A_y, \eta) = \lim_{p \to \infty} (\omega_t^{\partial_t \widehat{f}_p}(T_1), \omega_t^{\widehat{h}})_{L^2}, \tag{51}$$

for all $\widehat{h} \in \mathcal{F}^s([0, T_0])$. However, by Theorem 1.10, the existence of this limit can be verified if we are given $\mathcal{Z}^{T_2}$.

Conversely, assume that the limit (51) does exist for all $\widehat{h} \in \mathcal{F}^s([0, T_0])$. Then, by the Principle of Uniform Boundedness, the mappings

$$\eta \mapsto (\omega_t^{\partial_t \widehat{f}_p}(T_1), \eta)_{L^2}, \quad p \in \mathbb{Z}_+,$$

form a uniformly bounded family in the dual of $\mathcal{D}(\mathcal{M}_{\mathrm{e}}^s)$. By the Banach-Alaoglu theorem, we find a weak*-convergent subsequence

$$\omega_t^{\partial_t \widehat{f}_p}(T_1) \to A_y \in \Big(\mathcal{D}(\mathcal{M}_{\mathrm{e}}^s)\Big)',$$

which is the sought after limit distribution-form. □

Since $\mathrm{supp}(A_y)$ is a point, A_y consists of the Dirac delta and its derivatives. The role of the smoothness index, s is just to select the order of this distribution, as is seen in the following result.

Lemma 2.16 *Let $A_y = \lim_{p\to\infty} \omega_t^{\partial_t \widehat{f}_p}(T_1)$ is a distribution of order $s = 3$. Then A_y is of the form*

$$A_y(x) = (0, 0, d(\lambda \underline{\delta}_y(x)), 0), \tag{52}$$

where λ is a 1–form at y, $\lambda \in T_y^* M$. Furthermore, for any $\lambda \in T_y^* M$ there is a focusing source $\widetilde{f} =$ with A_y of form (52).

Proof: From the results in Section 2.5, we deduce that, when $s = 3$,

$$A_y \in \big(\mathcal{D}(\mathcal{M}_{\mathrm{e}}^3)\big)' \subset \mathbf{H}^{-3}.$$

Furthermore, from Remark 7, the electromagnetic waves (50) are of the form

$$\omega_t^{\partial_t \widehat{f}_p}(T_1) = (0, 0, d\eta_p, 0), \tag{53}$$

for some 1-forms η_p. Combining (these with the fact that $\mathrm{supp}(A_y) = \{y\}$, we see that $A_y = (0, 0, A_y^2, 0)$. Here A_y^2, expressed, for example, in Riemann normal coordinates (x^1, x^2, x^3) near y, must be of the form

$$A_y^2(x) = a^j \underline{\delta}_y(x)\theta_j + b^{jk}\partial_k \underline{\delta}_y(x)\theta_j,$$

where $\theta_j = (1/2)e_{jk\ell}dx^k \wedge dx^\ell$ and $e_{jk\ell}$ is the totally antisymmetric permutation symbol. Furthermore, by (53),

$$dA_y^2 = (a^j \partial_j \underline{\delta}_y(x) + b^{jk}\partial_k\partial_j \underline{\delta}_y(x))dV_g = 0.$$

Let φ be a compactly supported test function and, in the vicinity U of y,

$$\varphi(x) = x^j, \quad j = 1, 2, 3.$$

It follows that

$$0 = (dA_y^2, \varphi) = a^j.$$

Further, let ψ be a compactly supported test function and, in the vicinity U of y,

$$\psi(x) = x^j x^k, \quad j,k = 1,2,3.$$

As before, we obtain

$$0 = (dA_y^2, \psi) = b^{jk} + b^{kj}.$$

Thus, b^{jk} may be represented as $b^{jk} = e^{jk\ell}\lambda_\ell$, $\lambda_\ell \in T_y^* M$, implying that

$$A_y^2(x) = e^{jk\ell}\lambda_\ell \partial_k \underline{\delta}_y(x)\theta_j. \tag{54}$$

By the properties of the permutation symbols, $e^{jk\ell}$,

$$e^{jk\ell}\theta_j = \frac{1}{2} e^{jk\ell} e_{jpq} dx^p \wedge dx^q = \delta_p^k \delta_q^\ell dx^p \wedge dx^q.$$

Substituting this expression back to (54), we finally obtain

$$A_y^2(x) = \lambda_\ell \partial_k \underline{\delta}_y(x) dx^k \wedge dx^\ell = d(\underline{\delta}_y(x)\lambda_\ell dx^\ell),$$

as claimed. □

By the above results, for any $y \in M^{int}$ and $\lambda \in T_y^* M$, we can, in principle, find focusing sequences $\widetilde{f}$ such that $\omega_{tt}^{\widetilde{f}}(T_1) = A_y$, where A_y is of form (52). We should, however, stress that, at this stage, we can not control the corresponding $\lambda = \lambda(y)$.

Consider now a family of focusing sources $\widetilde{f}_y$, $y \in M^{\rm int}$, with the corresponding 1−forms $\lambda(y)$.

Lemma 2.17 *Given the admittance map $\mathcal{Z}^{T_2}$, it is possible to determine whether the map $y \mapsto \lambda_y$ is a nowhere vanishing 1-form valued C^∞–function.*

Proof: Let $\varphi \in \Omega^1 M^{\rm int}$ be an arbitrary compactly supported test 1−form. By Theorem 2.8, there is a generalized source $\widehat{h} \in \mathcal{F}^\infty$ such that

$$(\omega_t^{\widehat{h}})^1(T_1) = \varphi.$$

Let $\widetilde{f} = (\widehat{f}_p), p = 1,2,\cdots$, be a focusing source of order $s = 3$. Then, by Lemma 2.16 and the definition of the focusing sources, we have

$$\begin{aligned}\lim_{p\to\infty}(\omega_{tt}^{\widehat{f}_p}(T_1), \omega^{\widehat{h}}(T_1)) &= (A_y, \omega^{\widehat{h}}(T_1)) \\ = (d(\lambda\underline{\delta}_y), (\omega^{\widehat{h}})^2(T_1)) &= \int_M \lambda_y \underline{\delta}_y \wedge *\delta(\omega^{\widehat{h}})^2(T_1).\end{aligned} \tag{55}$$

Further, by Maxwell's equations,

$$\begin{aligned}\lambda_y \wedge *\delta(\omega^{\widehat{h}})^2(T_1) &= \lambda_y \wedge *(\omega_t^{\widehat{h}})^1(T_1)\\ &= \lambda_y \wedge *\varphi(y),\end{aligned}$$

Here, for $\lambda,\ \eta \in T_y^*M$,

$$\lambda \wedge *\eta = \langle \lambda, \eta\rangle_y \, dV_g = g^{jk}\lambda_j \eta_k \, dV_g.$$

Thus,

$$\lim_{p\to\infty} (\omega_{tt}^{\widehat{f}_p}(T_1), \omega^{\widehat{h}}(T_1)) = \langle \lambda,\ \varphi(y)\rangle_y. \tag{56}$$

By Theorem 1.10, the inner products on the left side of equation (56) are obtainable from the boundary data. Thus, we can find the map $y \to \langle \lambda, \varphi(y)\rangle_y$, $y \in M^{\rm int}$. Since $\varphi \in \Omega^1 M^{\rm int}$ is arbitrary, this determines whether $\lambda \in \Omega^1 M^{\rm int}$. It also determines whether $\lambda_y = 0$ or not for any $y \in M^{\rm int}$. This yields the claim. □

Another way to look at Lemma 2.17 is that the admittance map, $\mathcal{Z}^{T_2}$ determines, for any boundary source $h \in C_0^\infty(]0, T_0[; \Omega^1 \partial M)$, the values, at any $y \in M^{\rm int}$, of $\langle \lambda, (\omega_{tt}^h)^1(t)\rangle_y$ for some unknown $\lambda \in \Omega^1 M^{\rm int}$ and $T_1 < t < T_2 - \operatorname{diam}(M)$. Moreover, using this map, we can verify that the 1−forms $\lambda_k(y)$, corresponding to three families of focusing sources $\widetilde{f}_k(y)$, $k = 1, 2, 3$, are linearly independent at any $y \in M^{\rm int}$. These give rise to the following result.

Lemma 2.18 *Let $\mathcal{Z}^{T_2}$ be the admittance map of the Riemannian manifold with impedance (M, g, α). Then, for $T_1 \le t \le T_2 - \operatorname{diam}(M)$ and $\widehat{h} \in \mathcal{F}(]0, T_0[)$, it is possible to find the forms*

$$L(y)(\omega_t^{\widehat{h}}(y,t))^1, \quad K(y)(\omega_t^{\widehat{h}}(y,t))^2,$$

at any $y \in M^{\rm int}$. Here $L(y) : T_y^*M \to T_y^*M$ and $K(y) : \wedge^2 T_y^*M \to \wedge^2 T_y^*M$ are smooth sections of $\operatorname{End}(T^*M^{\rm int})$ and $\operatorname{End}(\Lambda^2 T^*M^{\rm int})$, correspondingly.

We emphasize that, at this stage, $L(y)$ and $K(y)$ are unknown. However, they are independent of t or $\widehat{h}$.

Proof: As M is already found, we can choose three differential 1−forms, $\xi_k \in \Omega^1 M^{\rm int}$ which, at any $y \in M^{\rm int}$, form a basis in T_y^*M. Using the families $\widetilde{f}_k(y)$ of focusing sources introduced earlier, we can construct, for any $\widehat{h} \in \mathcal{F}^\infty(]0, T_0[)$, the differential 1− form,

$$\rho^{\widehat{h}}(y,t) := \langle \lambda_k(y),\ (\omega_t^{\widehat{h}}(y,t))^1\rangle_y \, \xi_k(y). \tag{57}$$

This defines a smooth section, $L(y)$ of $\operatorname{End}(T^*M^{\rm int})$,

$$L(y)(\omega_t^{\widehat{h}}(y,t))^1 = \rho^{\widehat{h}}(y,t) \in \Omega^1 M^{\rm int},$$

proving the assertion for $(\omega_t^{\widehat{h}}(y,t))^1$ with $\widehat{h} \in \mathcal{F}^\infty(]0,T_0[)$. Its extension to $\widehat{h} \in \overline{\mathcal{F}(]0,T_0[)}$ is an immediate corollary of the fact that $\mathcal{F}^\infty(]0,T_0[)$ is dense in $\overline{\mathcal{F}(]0,T_0[)}$ in the $\mathcal{F}-$norm.

To analyse $(\omega_t^{\widehat{h}}(y,t))^2$, consider the form

$$\eta = (\frac{1}{\alpha} * \omega^3, \frac{1}{\alpha} * \omega^2, \frac{1}{\alpha} * \omega^1, \frac{1}{\alpha} * \omega^0) \tag{58}$$

(cf. ν-forms in formulae (9) and (12)). This form satisfies the complete Maxwell system

$$\eta_t + \widetilde{\mathcal{M}}\eta = 0,$$

where $\widetilde{\mathcal{M}}$ is the differential expression (15), corresponding to the manifold (M, g, α^{-1}). Then the admittance map $\mathbf{t}\eta^1|_{\partial M\times]0,T_2[} \to \mathbf{n}_{\alpha^{-1}}\eta^2|_{\partial M\times]0,T_2[}$ is the inverse of the given admittance map $\mathcal{Z}^{T_2} : \mathbf{t}\omega^1|_{\partial M\times]0,T_2[} \to \mathbf{n}_\alpha\omega^2|_{\partial M\times]0,T_2[}$. Thus, $\mathcal{Z}^{T_2}$ determine the admittance map, $\widetilde{\mathcal{Z}}^{T_2}$ for $\widetilde{\mathcal{M}}$ and we can apply the results for $(\omega_t^{\widehat{h}})^1$ to $(\eta_t^{\widehat{f}})^1$, where $\widehat{f} = \mathcal{Z}^{T_2}\widehat{h}$. Namely, we can find $\widetilde{L}(y)(\eta_t(y,t))^1$, where $L(y) : T_y^*M \to T_y^*M$ is a smooth section of $\mathrm{End}(T^*M^{\mathrm{int}})$ which, at this stage, is unknown. At last, since

$$*\widetilde{L}(y)(\eta_t^{\widehat{f}}(y,t))^1 = K(y)(\omega_t^{\widehat{h}}(y,t))^2,$$

for some smooth section, $K(y)$ of $\mathrm{End}(\Lambda^2 T^*M^{\mathrm{int}})$, the assertion follows. □

We note for the further reference, that, similar to the case of the 1−forms, the construction of $K(y)$ involves a choice of three differential 2−forms, that we denote by $\mu_k \in \Omega^2 M^{\mathrm{int}}$, and three families of generalized sources $\widetilde{\kappa}_k(y)$ that satisfy

$$\omega_t^{\partial_t\widetilde{\kappa}_k}(T_1) = (0, \delta(\mu_k \underline{\delta}_y(x)), 0, 0), \quad \mu_k \in \Lambda^2 T_y^*M, \quad k = 1,2,3. \tag{59}$$

Below we call the generalized sources $\widetilde{f}_k(y)$ the focusing sources for 2–forms and $\widetilde{\kappa}_k(y)$ the focusing sources for 1–forms.

Before going to a detailed discussion in the next sections of the reconstruction of α, let us explain briefly the main outline of this construction. It follows from Lemma 2.18 that, using the admittance map $\mathcal{Z}$, we can find the electromagnetic waves $\omega_t^f(t)$, $T_1 < t < T_2 - \mathrm{diam}\,(M)$, up to unknown linear transformations, L and K. We observe that, by Theorem 2.8, for any basis $\xi_k(y)$, $k = 1,2,3$, there are families $\widetilde{f}_k(y)$ of focusing sources, such that the corresponding transformation L is just identity. Indeed, to achieve this goal, we should choose $\widetilde{f}_k(y)$ in such a manner that, at any $y \in M^{\mathrm{int}}$,

$$(\omega_{tt}^{\widetilde{f}_k(y)})^2(T_1) = d(\lambda_k(y)\underline{\delta}_y), \tag{60}$$

where $\lambda_k(y)$ is dual to $\xi_k(y)$,

$$\langle \lambda_k(y),\, \xi_j(y)\rangle_y = \delta_{kj}. \tag{61}$$

In the next sections we will identify conditions on $\widetilde{f}_k(y)$ and $\widetilde{\kappa}_k(y)$, verifiable in terms of $\mathcal{Z}$, which make L and K to be identities.

2.8 Reconstruction of the wave impedance

In the previous section, it was shown how to select a family $\widetilde{f}_y$, $y \in M^{\rm int}$ of focusing sequences such that the corresponding electromagnetic fields concentrate at $t = T_1$ at a single point, y,

$$\lim_{p\to\infty} \omega_{tt}^{f_p}(T_1) = (0, 0, d(\lambda \underline{\delta}_y), 0).$$

Here $\lambda = \lambda_y \in T_y^* M$ is yet unknown. Moreover, we can select this family in such a manner that, as a function of y, $\lambda_y \in \Omega^1 M^{\rm int}$. In particular, it means that for times $0 < t < t_y = \tau(y, \partial M)$, the electromagnetic wave defined as

$$G_{\rm e}(y) = G_{\rm e}(x, y, t) = G_{\rm e}[\lambda](x, y, t) = \lim_{p\to\infty} \omega_{tt}^{f_p}(t + T_1), \tag{62}$$

satisfies the initial-boundary value problem

$$\begin{aligned} (\partial_t + \mathcal{M}) G_{\rm e}(y) &= 0 \text{ in } M \times]0, t_y[, \\ \mathbf{t} G_{\rm e}(y) &= 0 \text{ in } \partial M \times]0, t_y[, \\ G_{\rm e}(y)|_{t=0} &= (0, 0, d(\lambda \underline{\delta}_y), 0). \end{aligned} \tag{63}$$

The solution to this problem is called the *electric Green's function.* We will use this solution to reconstruct the scalar wave impedance, α on M. We start with analysis of some properties of $G_{\rm e}$. To this end, we will represent $G_{\rm e}$ in terms of the standard Green's function, $G = G(x, y, t)$ for the wave equation on $1-$ forms. Thus, G is defined as the solution to the following initial-boundary value problem

$$\begin{aligned} (\partial_t^2 + d\delta + \delta d) G(x, y, t) &= (\partial_t^2 + \Delta_\alpha^1) G(x, y, t) = \lambda \underline{\delta}_y(x) \underline{\delta}(t) \text{ in } M \times \mathbb{R} \\ \mathbf{t} G(x, y, t) = &\ 0, \\ G(x, y, t)|_{t<0} &= 0, \end{aligned} \tag{64}$$

where $\lambda \in T_y^* M$ is a given 1–form.

This Green's function has the following asymptotic behaviour.

Lemma 2.19 *For* $0 < t < t_y$*, Green's function* $G(x, y, t)$ *for the 1–form wave equation (64), has the representation*

$$G(x, y, t) = \underline{\delta}(t - \tau(x, y)) Q(x, y) \lambda + r(x, y, t).$$

Here $Q(x, y) : T_y^* M \to T_x^* M$ *is a bijective map that corresponds to a* $(1, 1)-$ *tensor depending smoothly on* $(x, y) \in M^{\rm int} \times M^{\rm int} \setminus diag(M^{\rm int})$*. The remainder* $r(x, y, t)$ *is a bounded function, when* $t < t_y$*, where* t_y *is small enough.*

The proof of this lemma is postponed in the Appendix.

In the following, we fix $y \in M^{\rm int}$ and $\lambda = \lambda_y \in T^*_y M$. By operating with the exterior derivative d on the both sides of the differential equation in (64), we see that

$$(\partial_t^2 + \Delta_\alpha^2) dG(x,y,t) = d(\lambda \underline{\delta}_y)\underline{\delta}(t).$$

Hence, using the decomposition $(\partial_t^2 + \mathbf{\Delta}) = (\partial_t + \mathcal{M})(\partial_t - \mathcal{M})$, we find that the form $\omega(t) = (0,0,dG(x,y,t),0)$ satisfies the equation

$$(\partial_t + \mathcal{M})\Big((\partial_t - \mathcal{M})(0,0,dG(x,y,t),0)\Big) = D_{y,\lambda}\underline{\delta}(t),$$

where

$$D_{y,\lambda} = (0,0,d(\lambda\underline{\delta}_y),0).$$

Let $\widetilde{G}_{\rm e}(y) = \widetilde{G}_{\rm e}(x,y,t)$ be defined as

$$\begin{aligned}\widetilde{G}_{\rm e}(x,y,t) &= (\partial_t - \mathcal{M})(0,0,dG(x,y,t),0) \\ &= (0,\delta dG(x,y,t),\partial_t dG(x,y,t),0).\end{aligned}$$

Then, due to the finite propagation speed, $G(y)|_{\partial M\times]0,t_y[} = 0$, so that $\widetilde{G}_{\rm e}(x,y,t)$ satisfies the boundary condition $\mathbf{t}\widetilde{G}_{\rm e}(y) = 0$ for $t < t_y$. Invoking the uniqueness of solution for (63), we see that $\widetilde{G}_{\rm e}(y) = G_{\rm e}(y)$, $t < t_y$.

Now, using Lemma 2.19, we obtain that

$$dG^1_e(x,y,t) = (Q(x,y)\lambda_y \wedge d\tau(x,y))\underline{\delta}^{(1)}(t-\tau(x,y)) + r_1(x,y,t),$$

where $\underline{\delta}^{(1)}$ is derivative of the delta-distribution and the residual r_1 is sum of the delta-distribution on $\partial B_y(t)$, where $B_y(t)$ is the ball of radius t centered in y, and a bounded function. Thus, we see that

$$\begin{aligned}\partial_t dG^1_{\rm e}(x,y,t) &= (Q(x,y)\lambda_y \wedge d\tau(x,y))\underline{\delta}^{(2)}(t-\tau(x,y)) + r_2(x,y,t), \\ \delta dG^1_{\rm e}(x,y,t) &= *\left(d\tau(x,y)\wedge *(Q(x,y)\lambda_y\wedge d\tau(x,y))\right)\underline{\delta}^{(2)}(t-\tau(x,y)) \\ &\quad + r_3(x,y,t), \qquad (65)\end{aligned}$$

where residuals r_2 and r_3 are sums of first and zeroth derivatives of the delta-distribution on $\partial B_y(t)$ and a bounded function. Moreover, by formulae (65),

$$\mathbf{t}_{B_y(t)}[Q(x,y)\lambda_y \wedge d\tau(x,y)] = 0, \qquad (66)$$

$$\mathbf{n}_{B_y(t)}\left[*\left(d\tau(x,y)\wedge *(Q(x,y)\lambda_y \wedge d\tau(x,y))\right)\right] = 0, \qquad (67)$$

where $\mathbf{t}_{B_y(t)}\omega^k$, $\mathbf{n}_{B_y(t)}\omega^k$ are the tangential and normal components of ω^k on $\partial B_y(t)$. This corresponds to the physical fact that the wavefronts of the electric and magnetic fields are perpendicular to the propagation direction. Let now $\widetilde{f}_y$ be a focusing source for a point $y \in M^{\rm int}$. Due to the definition (51) and the definition of the generalized source (39), there is a sequence $(f^y_p)_{p=1}^\infty$, $f^y_p \in C_0^\infty(]0,T_0[;\Omega^1\partial M)$, $p = 1,2,\cdots$ such that

$$\omega_t^{\widetilde{f}_y}(T_1) = \lim_{p\to\infty} \omega_t^{f^y_p}(T_1),$$

and the right side is understood in the sense of the distribution-forms on $\mathbf{\Omega}M^{\rm int}$. Then, for $t \geq 0$,

$$\omega_t^{\widetilde{f}_y}(t+T_1) = \lim_{p\to\infty} \omega_t^{f_p^y}(t+T_1). \tag{68}$$

Applying Lemma 2.18, it is possible to find, via given $\mathcal{Z}^{T_2}$, the magnetic components $K(x)(\omega_t^f(x,t+T_1))^2$, $f = f_p^y$ of these fields with K being a smooth section of $\mathrm{End}(\Lambda^2 T^* M^{\rm int})$. At last, using (68), we find

$$K(x)(\omega_t^{\widetilde{f}_y}(x,t+T_1))^2 = \lim_{p\to\infty} K(x)(\omega_t^{f_p^y}(x,t+T_1))^2 \in \mathcal{D}'(\Omega^2 M^{\rm int}).$$

Since

$$\omega_{tt}^{\widetilde{f}_y}(T_1) = (0,0,d(\lambda\underline{\delta}_y),0),$$

we see that

$$\omega_{tt}^{\widetilde{f}_y}(t+T_1) = G_e(\cdot\,,y,t),$$

when $t < t_y$. In particular, we can find the singularities of Green's function up to a linear transformation $K(x)$.

Hence we have shown:

Lemma 2.20 *Let $\widetilde{f}_y = (f_p^y)$, $p = 1,2,\cdots$, be a focusing source for a point y. Then, given the admittance map, $\mathcal{Z}^{T_2}$, it is possible to find the distribution $2-$form $KG_{\rm e}(y)^2$ for all x satisfying $\tau(x,y) < \widehat{t}_y$, where $\widehat{t}_y$ is small enough. In particular, the leading singularity of this form determine the $2-$form*

$$K(x)(Q(x,y)\lambda_y \wedge d\tau(x,y)). \tag{69}$$

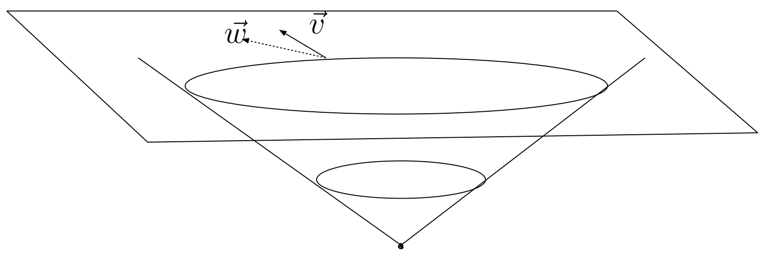

Figure 6: Vector $\vec{v}$ is the right singularity of the electromagnetic wave in the plane $M \times \{t\}$. The reconstructed singularity $\vec{w}$ has wrong direction, if the transformation matrix $K(x)$ is not isotropic.

As shown at the end of the previous section, $K \in \mathrm{End}(\Lambda^2 T^* M^{\rm int})$ was obtained by using three focusing sources $\widetilde{\kappa}_k(x)$, $k = 1,2,3$. Our next goal is to formulate conditions, verifiable using boundary data, for K to be isotropic, i.e.

$$K(x) = c(x)I, \tag{70}$$

with $c(x)$ being a smooth scalar function.

We start with observation that, for a given $\widetilde{\kappa}_k(x)$, $k = 1, 2, 3$, and any $\widetilde{f}_y$, we can find $\mathbf{t}_{B_y(t)} K(\cdot)(Q(\cdot, y)\lambda \wedge d\tau(\cdot, y))|_x$, for $x \in \partial B_y(t)$ and small $t > 0$. Here $K(x)$ is the linear transformation corresponding to the chosen $\widetilde{\kappa}_k(x)$. This follows from Lemma 2.20 and the fact that the underlying Riemannian manifold (M, g) is already found. When K is isotropic, it follows from (66) that

$$\mathbf{t}_{B_y(t)} K(\cdot)(Q(\cdot, y)\lambda \wedge d\tau(\cdot, y)) = 0. \tag{71}$$

Let us show that the condition (71), which is verifiable via boundary data, actually guarantees that K is of form (70).

Indeed, for a given $\lambda \in T^*_y M$ and any x, t with sufficiently small $t = \tau(x, y)$, (71) means that $K(x)(Q(x, y)\lambda \wedge d\tau(x, y))$ continues to be normal to the $2-$dimensional subspace, $T_x(\partial B_y(t)) \subset T_x M$, i.e.

$$[K(x)(Q(x, y)\lambda \wedge d\tau(x, y))] (X, Y) = 0, \quad X, Y \in (T_x \partial B_y(t)).$$

As $Q(x, y) : T^*_y M \to T^*_x M$ is bijective, $Q(x, y)\lambda \wedge d\tau(x, y)$ runs through the whole $2-$dimensional subspace in $\Lambda^2 T^*_x M$, normal to $T_x(\partial B_y(t))$, when λ runs through $T^*_y M$. Therefore, (71) implies that $K(x)$ keeps this subspace invariant.

Let us now vary y and t keeping x fixed. Then $T_x(\partial B_y(t))$ runs through the whole Grassmannian manifold, $G_{3,2}(T_x M)$. Therefore, (71) implies that, if $\omega^2 \in \Lambda^2 T^*_x M$ is normal to a subspace $\mathcal{L} \in G_{3,2}$, then $K(x)\omega^2$ remains to be normal to $\mathcal{L}$. This is implies that the eigenspace of $K(x)$ has dimension three and hence we must have

$$K(x) = c(x)I,$$

where $c(x)$ is a scalar function and I is the identity map in $\Lambda^2 T^*_x M$.

In the following, we always take focusing sources which satisfy (70).

Thus, we can find the values of the 2-forms

$$(\widetilde{\omega}_t^{\widehat{f}}(x, T_1)^2 =_{def} c(x)(\omega_t^{\widehat{f}}(x, T_1))^2, \quad \widehat{f} \in \mathcal{F}.$$

Our further considerations are based on the equation,

$$d(\omega_t^{\widehat{f}}(T_1)^2) = 0. \tag{72}$$

Thus, we intend to choose those focusing sources $\widetilde{\kappa}_k(x)$ which produce $K(x) = c(x)I$ with

$$d(\widetilde{\omega}_t^{\widehat{f}}(T_1))^2 = 0, \quad \widehat{f} \in \mathcal{F}^\infty.$$

In this case,

$$0 = d(c(x)(\omega_t^{\widehat{f}} T_1))^2) = dc(x) \wedge (\omega_t^{\widehat{f}}(T_1))^2,$$

due to (72). By the global controllability, the form $(\omega_t^{\widehat{f}})^2(x, T_1)$ runs through the whole $\Lambda^2 T_x^* M$. This implies that $dc(x) = 0$ at any $x \in M^{\text{int}}$. Hence, $c(x) = c_0$ is a constant. Thus, we choose focusing sources, $\widetilde{\kappa}_k(x)$, so that

$$(\widetilde{\omega}_t^{\widehat{h}}(T_1))^2 = c_0(\omega_t^{\widehat{h}}(T_1))^2.$$

Evaluating the inner products,

$$\int_M \widetilde{\omega}_t^2(x, T_1) \wedge *\widetilde{\omega}_t^2(x, T_1) = \int_M c_0\omega_t^2(x, T_1) \wedge *c_0\omega_t^2(x, T_1),$$

we can compare them with the energy integrals,

$$\int_M \frac{1}{\alpha(x)} \omega_t^2(x, T_1) \wedge *\omega_t^2(x, T_1),$$

which can be found from the boundary data by means of Theorem 1.10 . By considering waves $\omega_t^{\widehat{f}_j}(x, T_1)$ with

$$\lim_{j\to\infty} \text{supp}(\omega_t^{\widehat{f}^j}(\cdot, T_1)) = \{y\},$$

we find the ratio

$$\lim_{j\to\infty} \frac{\int_M (\widetilde{\omega}_t^{\widehat{f}_j}(x, T_1))^2 \wedge *(\widetilde{\omega}_t^{\widehat{f}_j}(x, T_1))^2}{\int_M \frac{1}{\alpha(x)} (\omega_t^{\widehat{f}_j}(x, T_1))^2 \wedge *(\omega_t^{\widehat{f}_j}(x, T_1))^2} = c_0^2 \alpha(y).$$

The above considerations imply that, using $\mathcal{Z}^{T_2}$, we can determine $\alpha(x)$ up to a constant c_0^2. Since the impedance map satisfies $\mathcal{Z}_{M,g,c\alpha} = c^{-1}\mathcal{Z}_{M,g,c\alpha}$, we see that, knowing the impedance map, we can also determine c_0.

As the fact that $\mathcal{Z}^T$, $T \geq 4\,\text{diam}M$ determines (M, g) is already proven, this completes the proof of Theorem 2.1. □

2.9 Back to $\mathbb{R}^3$

In this section we use the obtained uniqueness result for Maxwell equation on a 3-dimensional manifold to analyze to the group of transformations which preserve the boundary data in the dynamical inverse problem for Maxwell's equation (1) in a domain $\Omega \subset \mathbb{R}^3$.

Assume that two Maxwell systems with electric and magnetic permittivities $\epsilon_k^j(x)$, $\mu_k^j(x)$ and $\widetilde{\epsilon}_k^j(x)$, $\widetilde{\mu}_k^j(x)$, $x \in \Omega$, have the same admittance map $\mathcal{Z}^T$ on $\partial\Omega \times [0,T]$, where T is sufficiently large. Denote by (M, g, α) and $(\widetilde{M}, \widetilde{g}, \widetilde{\alpha})$ the corresponding abstract Riemannian manifolds with impedance. By Theorem 2.1,

$$(\widetilde{M}, \widetilde{g}, \widetilde{\alpha}) = (M, g, \alpha)$$

i.e., there is an isometry $H : (M, g) \to (\widetilde{M}, \widetilde{g})$ and $\alpha = H_* \widetilde{\alpha}$. We can represent the abstract manifold (M, g, α) as the domain Ω with the metric tensor, $g_{ij}(x)$, given in Euclidean coordinates by (6), (8), and scalar impedance, $\alpha(x)$ in these coordinates. Similarly, we represent manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\alpha})$ using Euclidean coordinates in Ω and obtain the metric tensor $\widetilde{g}_{ij}$ and impedance $\widetilde{\alpha}$. Then $H : \widetilde{M} \to M$ corresponds to a diffeomorphism,

$$\widetilde{X} : \Omega \to \Omega, \quad \widetilde{X}|_{\partial\Omega} = id|_{\partial\Omega}, \tag{73}$$

and

$$\widetilde{g} = \widetilde{X}_* g, \quad \text{i.e.,} \quad \widetilde{g}^{ij}(\widetilde{x}) = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} g^{pq}(x), \quad \widetilde{x} = \widetilde{X}(x), \tag{74}$$
$$\widetilde{\alpha} = \widetilde{X}_* \alpha, \quad \text{i.e.,} \quad \widetilde{\alpha}(\widetilde{x}) = \alpha(x).$$

Using (74) and (8), we see also that

$$\widetilde{g}_\epsilon = \widetilde{X}_* g_\epsilon, \quad \text{i.e.,} \quad \widetilde{g}_\epsilon^{ij}(\widetilde{x}) = \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} g_\epsilon^{pq}(x), \quad \widetilde{x} = \widetilde{X}(x). \tag{75}$$

Employing formula (6), we obtain

$$\epsilon_q^p = \sqrt{g_\epsilon} g_\epsilon^{pr} \delta_{pq}, \quad \widetilde{\epsilon}_q^p = \sqrt{\widetilde{g}_\epsilon} \widetilde{g}_\epsilon^{pr} \delta_{pq}. \tag{76}$$

Combining formulae (74)–(76) and introducing

$$\epsilon^{pq} = \epsilon_r^p \delta^{jq}, \quad \widetilde{\epsilon}^{pq} = \widetilde{\epsilon}_r^p \delta^{jq},$$

we obtain

$$\widetilde{\epsilon}^{pq} = \frac{1}{\text{Det}\,(D\widetilde{X})} \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} \epsilon^{pq}(x), \quad \widetilde{x} = \widetilde{X}(x). \tag{77}$$

Similarly,

$$\widetilde{\mu}^{pq} = \frac{1}{\text{Det}\,(D\widetilde{X})} \frac{\partial \widetilde{x}^i}{\partial x^p} \frac{\partial \widetilde{x}^j}{\partial x^q} \mu^{pq}(x), \quad \widetilde{x} = \widetilde{X}(x) \tag{78}$$

Clearly, if $\widetilde{X} : \Omega \to \Omega$ and $\widetilde{X}|_{\partial\Omega} = id$, the admittance map $\mathcal{Z}^T$, $T > 0$ is preserved in transformations (73), (77), (78).

Thus we have proven the following result.

Theorem 2.21 *The group of transformations for Maxwell's equations (1)–(4) with a scalar wave impedance, which preserves the admittance map $\mathcal{Z}^T$, $T > 4\,\mathrm{diam}(M, g)$, is generated by the group of diffeomorphisms of Ω satisfying (73). The corresponding transformations of ϵ and μ are then defined by formulae (77), (78).*

Remark 9. It follows from (77), (78), that ϵ^{jk} and μ^{jk} do not transform like tensors. This is due to the special role played by the underlying Euclidean metric $g_0^{ij} = \delta^{ij}$, which does not change by diffeomorphisms $\widetilde{X}$. It should be noted that this form of transformations is observed also in the study of the Calderón inverse conductivity problem. Indeed, it is shown in [53] that, for the conductivity equation in $\Omega \subset \mathbb{R}^2$, the boundary measurements determine the anisotropic conductivity up to same group of transformations as described in Theorem 2.21. The Calderon problem is closely related to the inverse problem for Maxwell's equation, for instance, the low-frequency limit of the admittance map $\mathcal{Z}^\infty$ is related to the Dirichlet-to-Neumann map for the conductivity equation [38].

2.10 Outlook

There are several direction to which the present work can be extended.

1. Natural question is the minimal observation time required to parameter reconstruction. It can be shown that the admittance map $\mathcal{Z}^t$ for any $t > 0$. Thus, it follows from the above, that $\mathcal{Z}^T$, $T > 2\,\mathrm{rad}M$ determines uniquely the manifold M, metric g and wave impedance α. The reconstruction of $\mathcal{Z}^t$ for any $t > 0$ may be obtained by a direct continuation of the admittance map, i.e., without solving the inverse problem. This continuation is a direct generalization to the considered Maxwell's case of the technique developed in [31], [25] for the scalar wave equation. An analogous method has recently be applied to Maxwell's equations in [10].

2. Another natural inverse boundary value problem is the inverse boundary spectral problem for the electric Maxwell operator $\mathcal{M}_\mathrm{e}$ defined in Definition 1.3. The problem is to determine the metric g and wave impedance α, or, in the other words, ε and μ from the non-zero eigenvalues λ_j of $\mathcal{M}_\mathrm{e}$ and the normal boundary values of the corresponding eigenforms. This problem was studied in, e.g. [36], [37], for scalar Maxwell's equations. For the considered anisotropic case, this requires significant modifications of the method developed in this paper and will be published elsewhere.

3. It often occurs in applications that the measurements are made only on a part of boundary. In formalism of this paper this means that we actually know only the restriction of the admittance map to the part $\Gamma \times [0, T]$ of the lateral boundary, i.e. we are given

$$\mathcal{Z}^T f|_{\Gamma\times[0,T]}, \quad f \in C_0^\infty([0,T]; \Omega^1\Gamma).$$

For the scalar wave equation the corresponding problem is studied in [23], [24] (see also [33], [25], [26]). The combination of these methods and those of the present paper will be useful for analyzing the corresponding problem for Maxwell's equations.

Appendix: The WKB approximation

Here we consider asymptotic results for Green's function or, more precisely, for Green's 1-form, $G(x,y,t) = G_\lambda(x,y,t)$, which is defined as the solution for the wave equation

$$\begin{aligned} &\partial_t^2 G_\lambda + (d\delta + \delta d) G_\lambda = a\underline{\delta}_y(x)\underline{\delta}(t) \quad \text{in } M \times \mathbb{R}_+, \\ &G_\lambda(\cdot, y, t) = 0 \quad \text{for } t < 0, \quad \mathbf{t}G_\lambda(\cdot, y, t) = 0, \end{aligned} \tag{79}$$

where $\delta = \delta_\alpha$. Here, λ is a 1-form $\lambda = \sum_{i=1}^3 \lambda_i \, dx^i$ in normal coordinates $(B_y(\rho), X)$, $X = (x^1, x^2, x^3)$, near a point $y \in M^{\text{int}}$, $X(y) = 0$. We assume that $B_y(\rho) \cap \partial M = \emptyset$. In these coordinates, $\underline{\delta}_y(x) = \underline{\delta}(x)$, when $x \in U$. Clearly, we can find, instead of the solution to (79), the fundamental solution

$$\begin{aligned} &\partial_t^2 G + (d\delta + \delta d) G = I\underline{\delta}(x)\underline{\delta}(t) \quad \text{in } M \times \mathbb{R}_+, \\ &G(\cdot, y, t) = 0 \quad \text{for } t < 0, \quad \mathbf{t}G(\cdot, y, t) = 0, \end{aligned} \tag{80}$$

where I is the 3×3 identity matrix. Equation (80), written in normal coordinates, becomes a hyperbolic system

$$\left\{ (\partial_t^2 - g^{ij}\partial_i\partial_j) I + B^i \partial_i + C \right\} G = I\underline{\delta}(x)\underline{\delta}(t). \tag{81}$$

Here $g^{ij}(x)$ is the metric tensor in these coordinates with

$$g^{ij}(0) = \delta^{ij}, \; \partial_k g^{ij}(0) = 0, \tag{82}$$

and $B^i(x)$, $C(x)$ are smooth 3×3 matrices. We note that, in normal coordinates, $\tau(x,y) = |x|$. However, we prefer to keep the notation τ to stress the invariant nature of considerations below.

Following [16], [2], which deal with the scalar case, we search for the solution to (81) in the WKB form:

$$G(x,t) = G_0(x)\, \underline{\delta}(t^2 - \tau^2) + \sum_{l \geq 1} G_l(x)\, S_{l-1}(t^2 - \tau^2), \tag{83}$$

where $S_l(s) = s_+^l / \Gamma(l+1)$. Substitution of expression (83) into equation (81) gives rise to the recurrent system of (transport) equations. The principal one is the equation for G_0,

$$4\tau \frac{dG_0}{d\tau}(\tau\widehat{x}) + \left\{ (g^{ij}(\tau\widehat{x})\, \partial_i\partial_j \tau^2(\tau\widehat{x}) - 6)\, I + B^i(\tau\widehat{x})\, \partial_i \tau^2(\tau\widehat{x}) \right\} G_0(\tau\widehat{x}) = 0, \tag{84}$$

where $\widehat{x} = x/\tau$. In addition, to satisfy initial conditions $I\underline{\delta}(x)\underline{\delta}(t)$, corresponding to the right side in the wave equation (80), we require that

$$G_0(0) = \frac{1}{2\pi} I. \tag{85}$$

By (82), $g^{ij}\partial_i\partial_j\tau^2 - 6$ is a smooth function near $x = 0$ and $g^{ij}\partial_i\partial_j\tau^2|_{x=0} - 6 = 0$. Also, $\partial_i\tau^2|_{x=0} = 0$. Therefore,

$$\frac{1}{4\tau}\left\{(g^{ij}(\tau\widehat{x})\,\partial_i\partial_j\tau^2(\tau\widehat{x}) - 6)\,I + B^i(\tau\widehat{x})\,\partial_i\tau^2(\tau\widehat{x})\right\}$$

is a smooth function of $(\tau, \widehat{x})$, so that $G_0(x)$ is a smooth 3×3 matrix of x for $\tau > 0$.

Actually, the matrix $G_0(x)$ is smooth everywhere in the neighborhood of y, i.e. $x = 0$, including $x = 0$ itself. Indeed, if we write the Taylor expansion of $(g^{ij}(x)\partial_i\partial_j\tau^2(x) - 6)\,I + B^i(x)\partial_i\tau^2(x)$ near $x = 0$ and divide the result by $\tau = |x|$, we obtain that

$$(g^{ij}(x)\partial_i\partial_j\tau^2(x) - 6)\,I + B^i(x)\partial_i\tau^2(x) = \sum_{|\beta|\geq 1} D_\beta \tau^{|\beta|-1}\widehat{x}^\beta. \tag{86}$$

Substituting the Taylor expansion (with respect to τ) of $G_0(\tau, \widehat{x})$,

$$G_0(\tau, \widehat{x}) = \sum_{p\geq 0} G_{0,p}(\widehat{x})\tau^p,$$

into (84) and using (86), (85), we obtain that $G_{0,p}(\widehat{x})\tau^p$ are homogeneous polynomials of x of degree p:

$$G_{0,p}(\widehat{x})\tau^p = \sum_{|\beta|=p} G_{0,\beta}x^\beta.$$

Then, $G_0^p(x) = \sum_{|\beta|<p} G_{0,\beta}\, x^\beta$ satisfies

$$4\tau\frac{dG_0^p}{d\tau} + \left\{(g^{ij}\,\partial_i\partial_j\tau^2 - 6)\,I + B^i\,\partial_i\tau^2\right\} G_0^p = \theta^p, \tag{87}$$

where

$$\theta^p(\tau, \widehat{x}) = \theta^p(x) \in C^\infty(U), \quad \theta^p(x) = O(\tau^p). \tag{88}$$

We construct G_0 as $G_0^p\,(I + \widetilde{G}_0^p)$. Substituting this expression into (84) and using (87), (88), we obtain that

$$4\tau\frac{d\widetilde{G}_0^p}{d\tau}(\tau, \widehat{x}) = A^p(\tau, \widehat{x}) + A^p(\tau, \widehat{x})\,\widetilde{G}_0^p(\tau, \widehat{x}), \quad \widetilde{G}_0^p(0) = 0,$$

where

$$A^p(\tau, \widehat{x}) = -\left(G_0^p(\tau, \widehat{x})\right)^{-1}\,\theta^p(\tau, \widehat{x}).$$

Therefore, $A^p(\tau, \widehat{x}) \in C^\infty$ as a function of $(\tau, \widehat{x})$ and $A^p(\tau) = O(\tau^p)$. This implies that $\widetilde{G}_0^p(\tau, \widehat{x}) = O(\tau^p)$ and is C^∞ smooth as a function of $(\tau, \widehat{x})$, so that $\widetilde{G}_0^p$, considered as a function of $x = \tau\widehat{x}$ is in $C^p(B_y(\rho))$. As $p > 0$ is arbitrary and the solution G_0 of (84), (85) is unique, $G_0 \in C^\infty(U)$.

For G_l, $l \geq 1$, we obtain transport equations

$$4\tau \frac{dG_l}{d\tau} + \left\{ (4l - 6 + g^{ij}(x)\partial_i\partial_j\tau^2(x))\, I + B^i(x)\partial_i\tau^2 \right\} G_l$$
$$= \left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C \right] G_{l-1},$$

and $G_l(0) = 0$. If we write $G_l = G_0 F_l$, we obtain for F_l the equations

$$4\tau \frac{dF_l}{d\tau} + 4l\, F_l = G_0^{-1} \left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C \right] G_{l-1}, \quad F_l(0) = 0, \tag{89}$$

with their solutions

$$F_l(x) = \frac{1}{4}\tau^{-l} \int_0^\tau G_0^{-1}(s\widehat{x}) \left\{ \left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C \right] G_{l-1} \right\} (s\widehat{x})\, s^{l-1} ds, \tag{90}$$

being a smooth function of x.

As (81) is a hyperbolic system, it is easy to show that the right side of (83) represents the asymptotics with respect to smoothness of the Green's 1−form $G(x, y, t)$, when $t < \tau(y, \partial M)$. Clearly, (83) can also be written in the form

$$G(x, t) = G_0(x)\, \underline{\delta}(t^2 - \tau^2) + r(x, t)$$

where $r(x, t)$ is a bounded 3×3 matrix.

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