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ANALYSIS OF REDUCED FINITE ELEMENT SCHEMES IN PARAMETER DEPENDENT ELLIPTIC PROBLEMS

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Abstract: *This thesis presents an analysis of modified finite element schemes applied to parameter dependent elliptic problems prone to locking. Two different problems of similar type are considered: the problem of anisotropic heat conduction and the thin shell problem.*

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This thesis consists of the overview and the following publications.

[I] V. Havu, J. Pitkäranta: An Analysis of Finite Element Locking in a Parameter Dependent Model Problem, Helsinki University of Technology, Institute of Mathematics, Research Report A420 (1999). To appear in Numer. Math.

[II] V. Havu, J. Pitkäranta: Analysis of a Bilinear Finite Element for Shallow Shells I: Approximation of Inextensional Deformations, Helsinki University of Technology, Institute of Mathematics, Research Report A430 (2000). To appear in Math. Comp.

[III] V. Havu, J. Pitkäranta: Analysis of a Bilinear Element for Shallow Shells II: Consistency Error, Helsinki University of Technology, Institute of Mathematics, Research Report A433 (2001). Submitted to Math. Comp.

[IV] V. Havu: An Analysis of Asymptotic Consistency Error in a Parameter Dependent Model Problem, Helsinki University of Technology, Institute of Mathematics, Research Report A434 (2001). Submitted to Numer. Math.

[V] V. Havu, H. Hakula: An Analysis of a Bilinear Reduced Strain Element in the Case of an Elliptic Shell in a Bending Dominated State of Deformation in Advances in Computational Engineering & Sciences, Satya N. Atluri, Frederick W. Brust (eds.) (2000), pp. 732–737

In [I] the author conducted the detailed analysis, a portion of the writing and the numerical experiments. In [II] and [III] the author is responsible for carrying out the main part of the analysis and the writing. In [V] the author is responsible for all aspects of the work, except for the numerical results presented.

1 Introduction

The phenomenon of locking, or parametric error amplification, is a well-known problem arising when trying to solve parameter dependent elliptic problems with a low-order finite element method when the parameter value is close to a limit value. Typical problems suffering from locking include the plate-bending problem [19], the problem of anisotropic heat conduction [2], and perhaps the most challenging one: the shell problem of linear elasticity [16, 17].

Among the above three problems the anisotropic heat conduction and the shell problem are similar in that they both admit two different asymptotic states depending on the constraints imposed on the boundary of the domain. The varying behavior reflects the fact that for the same problem some physical quantities can be distributed in different ways in different situations. In the case of anisotropic heat conduction the heat flux is dominant in the direction of low conductivity in the "hot" state whereas in the "cool" state the main heat conduction occurs in the direction of high conductivity as one would normally expect.

In shells the deformation energy can be associated either to bending of the shell or to stretching of the shell membrane. In the former case the deformation state is called bending-dominated and in the latter membrane-dominated. It is also worth noting that some bending-dominated states are inextensional in the sense that all the deformation energy is due to bending of the shell and no stretching is present.

The "hot" and the bending-dominated states suffer from locking at parameter values close to the limit value whereas the "cool" and the membrane-dominated states can be resolved in an optimal way for the entire parameter range using the elementary finite element scheme. Typical examples of the "hot" and "cool" states are shown in Figure 1 where the only difference between the two figures is in the conditions imposed at the boundary of the domain. Figure 2 displays a bending-dominated state of deformation of a cylindrical shell. In the locking states, typically very little energy is associated to large amplitudes of the unknown quantities so that the standard finite element scheme – being an energy-based method – fails to pinpoint the essential characteristics of the solution thus leading to very unsatisfactory results.

Technically locking can be viewed as a problem of finding a suitable finite element approximation to a solution of a problem posed in $\mathcal{U}_0$, a closed subspace of the energy space $\mathcal{U}$. In an extreme case the intersection of the subspace $\mathcal{U}_0$ and the finite element space $\mathcal{U}_h$ may consist only of the zero function, so it is clear that locking can lead to a severe approximation failure if the classical finite element scheme is employed.

Probably the most common way to circumvent locking is to commit a variational crime on the associated bilinear form $\mathcal{A}(u, v)$ thus ending up with a modified, usually mesh dependent, form $\mathcal{A}_h(u, v)$. The aim of this procedure is to allow the solution to be sought for in a larger space $\mathcal{U}_{0,h} \subset \mathcal{U}_h$

which is not any more a subspace of the space $\mathcal{U}_0$. This approach for plates is discussed thoroughly in [19].

However, a modification to the original bilinear form entails a consistency error component that can be considered negligible in the "hot" or bending-dominated state but by no means in the "cool" or membrane-dominated state. The result is that one is faced with a conflict when trying to design a robust general-purpose low-order element for problems exhibiting such a dual character: On one hand the modification should be significant to remove the locking effect, on the other hand too large a modification causes a large consistency error component in the state free of locking. Given these conditions it can be anticipated that the path in between is narrow – if any exists.

In fact, the quest for simple and efficient low-order finite element, applicable to problems of both asymptotic types in the linear theory of thin, elastic shells has been going on for years. The most prominent elements belonging to the MITC-family [3] have been a subject to several studies [4, 5], and only recently it was shown [14] that this formulation is equivalent to a modified formulation of a classical shell model due to Reissner and Naghdi [15].

In this thesis both the problem of anisotropic heat conduction and the shell problem are studied within the context of bilinear elements. The former is regarded as a simple model problem capable of revealing the essential aspects associated with the use of reduced finite element schemes, whereas the main question posed for the latter problem is: Under which hypothesis – if any – the lowest order MITC-type element, MITC4, can serve as a general-purpose shell element? It turns out that in both cases it is necessary to make heavy assumptions on the mesh and on the regularity of the solution, but under these assumptions it can be shown that

- Locking can be avoided in the "hot" or bending-dominated state.
- The consistency error component in the "cool" or membrane-dominated state is of acceptable magnitude.

However, the behavior of the modified schemes under discussion in a more general setup and their response to boundary layers commonly present remains a wide open question available for further study.

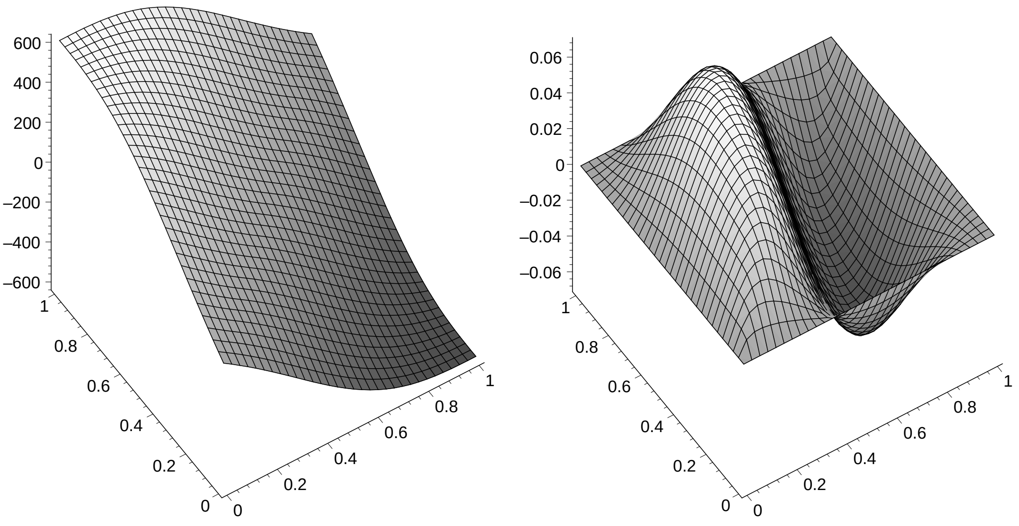
2 The phenomenon of locking

When the parameter ϵ in a parameter dependent elliptic problem approaches a certain limit value ϵ_0 difficulties in the finite element approximation of the solution may arise. In this case the variational problem: Find $u \in \mathcal{U}$ s.t.

$$\mathcal{A}(u, v) = \phi(v) \quad \forall v \in \mathcal{U}$$

turns into a constrained problem: Find $u_0 \in \mathcal{U}_0$ s.t.

$$\mathcal{A}(u, v) = \phi(v) \quad \forall v \in \mathcal{U}_0.$$



(a) “Hot” (b) “Cool”

Figure 1: “Hot” and “cool” states in the problem of anisotropic heat conduction. In the former the heat flux across the boundary vanishes whereas in the latter the temperature is zero on the boundary of the domain.

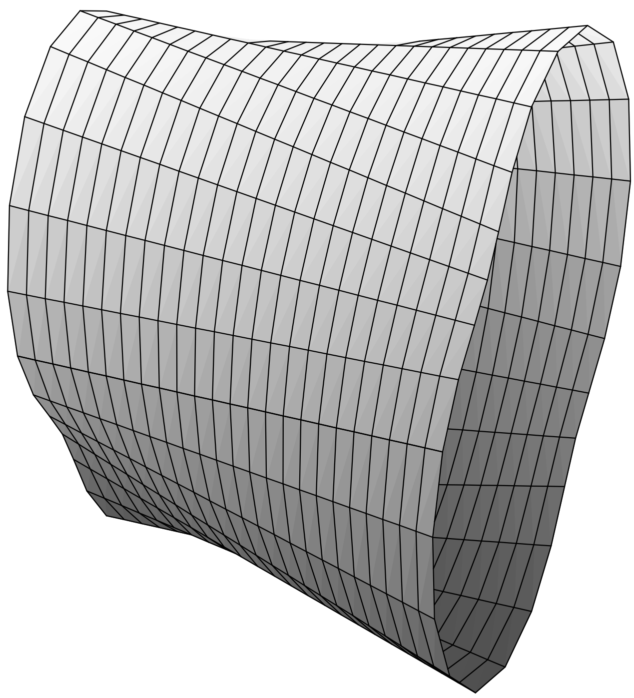


Figure 2: A bending-dominated deformation of a cylindrical shell.

It may very well then happen that $\mathcal{U}_0 \cap \mathcal{U}_h = \{\text{very small}\}$ thus preventing the normal convergence of the finite element approximation u_h to the exact solution u also for parameter values ϵ near ϵ_0. In fact, if a standard finite element method is used for solving such a problem the relative error e in the energy norm may behave

$$e \sim |\epsilon - \epsilon_0|^{-\alpha} h^p. \tag{1}$$

Here α denotes the severity of the locking and p is the degree of elements used in the approximation. For instance, in the case of thin shells the parameter is the thickness of the shell, t, the limit value is zero and $\alpha \in [0, 1]$ so that in the worst case $e \sim t^{-1} h^p$ and in the best case no locking is present, but the finite element scheme resolves the problem in an optimal way. In addition to these extreme possibilities the locking factors $\alpha = 1/2, 1/3, 1/4$ may be present depending on the shell geometry, the load, and on the constraints imposed at the boundaries [11, 17, 18].

The general result (1) indicates also that a high-order method may be used to improve the convergence properties of the finite element scheme [10, 12]. Indeed, in the worst case the mesh needs to be overrefined by a factor $\sim t^{-1/p}$ which becomes quite moderate even for small t when p is large enough. Also in some special cases a high order method may remove the parameter dependence of the error when a certain threshold value of p is exceeded [16]. The more general case of the hp-method in this context is discussed in [9, 20].

3 Error analysis principles

The idea of splitting the discretization error into two components, the approximation error and the consistency error, is not new. In fact, it dates back to the early days of mathematical finite element analysis [21]. Let us, however, briefly review the reasoning here. Assume that a variational problem: Find $u \in \mathcal{U}$ s.t.

$$\mathcal{A}(u, v) = \phi(v) \quad \forall v \in \mathcal{U} \tag{2}$$

is given in an energy space $\mathcal{U}$. Assume in addition that the finite element solution to (2) is given by: Find $u_h \in \mathcal{U}_h$ s.t.

$$\mathcal{A}_h(u_h, v) = \phi(v) \quad \forall v \in \mathcal{U}_h$$

for a finite element space $\mathcal{U}_h \subset \mathcal{U}$ and a modified bilinear form $\mathcal{A}_h(u, v)$. Let the error indicator be

$$e = \frac{|||u - u_h|||_h}{|||u|||}$$

where $|||\cdot||| = \sqrt{\mathcal{A}(\cdot,\cdot)}$ denotes the energy norm and $|||\cdot|||_h = \sqrt{\mathcal{A}_h(\cdot,\cdot)}$ is the modified energy norm. Then it is possible to split $u_h = \tilde{u}_h + z_h$ where $\tilde{u}_h$ is defined as the best approximation to u_h in $\mathcal{U}_h$ so that

$$\mathcal{A}_h(\tilde{u}_h, v) = \mathcal{A}_h(u, v) \quad \forall v \in \mathcal{U}_h$$

implying that in particular

$$\mathcal{A}_h(u - \tilde{u}_h, z_h) = 0.$$

This leads to an orthogonal splitting of the error

$$|||u - u_h|||_h^2 = |||u - \tilde{u}_h|||_h^2 + |||z_h|||_h^2$$

or

$$e^2 = e_A^2 + e_C^2$$

where the approximation error e_A is given by

$$e_A = \frac{|||u - \tilde{u}_h|||_h}{|||u|||}$$

and the consistency error e_C by

$$e_C = \frac{|||z_h|||_h}{|||u|||}.$$

It is also worth noting that for every $v \in \mathcal{U}_h$

$$\begin{aligned} \mathcal{A}_h(z_h, v) =& \mathcal{A}_h(u_h, v) - \mathcal{A}_h(\tilde{u}_h, v) \\ =& \mathcal{A}(u, v) - \mathcal{A}_h(u, v) \\ =& \phi(v) - \mathcal{A}_h(u, v) = \Psi_u(v) \end{aligned} \tag{3}$$

so that the consistency error is given by

$$e_C = \sup_{v \in \mathcal{U}_h, v \neq 0} \frac{(\mathcal{A} - \mathcal{A}_h)(u, v)}{|||u|||\ |||v|||_h}$$

and the element $z_h \in \mathcal{U}_h$ giving the value of e_C can be solved from (3) if the exact solution $u \in \mathcal{U}$ is known.

The above reasoning extends to the case of the limit parameter value ϵ_0. For $u_0 \in \mathcal{U}_0$ the approximation error is given by

$$e_A = \inf_{v \in \mathcal{U}_{0,h}} \frac{|||u_0 - v|||_h}{|||u_0|||}$$

and the asymptotic consistency error by

$$e_C = \sup_{v \in \mathcal{U}_{0,h}, v \neq 0} \frac{(\mathcal{A} - \mathcal{A}_h)(u_0, v)}{|||u_0|||\ |||v|||_h}.$$

4 A simple model problem

A simple yet interesting model problem on the phenomena associated with the use of low-order finite element method in elliptic parameter dependent problems is the problem of anisotropic heat conduction. The setup was first discussed in [2] where the failure of the standard finite element scheme was noted. The problem is given on the unit square $\Omega = (0,1) \times (0,1)$ as

$$-\frac{\partial^2 u}{\partial \xi^2} - \epsilon^2 \frac{\partial^2 u}{\partial \eta^2} = f \text{ in } \Omega \tag{4}$$

with

$$\begin{cases} \xi = \alpha x + \beta y \\ \eta = -\beta x + \alpha y \end{cases}$$

for some $\alpha, \beta \neq 0$, $\alpha^2 + \beta^2 = 1$ and $f \in L^2(\Omega)$ or $f \in \mathcal{D}'(\Omega)$. To get a full view on the spectrum of the problem it is necessary to consider at least two different boundary conditions:

A. $u = 0$ on $\partial\Omega$

C. $\partial_\nu u = 0$ on $\partial\Omega$.

Here

$$\partial_\nu u = (\alpha n_x + \beta n_y)\frac{\partial u}{\partial \xi} + \epsilon^2(-\beta n_x + \alpha n_y)\frac{\partial u}{\partial \eta}$$

(n_x, n_y) being the outward unit normal to $\partial\Omega$. The equation (4) leads to the variational formulation: Find $u \in \mathcal{U}$ s.t.

$$\mathcal{A}(u,v) =< \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} > +\epsilon^2 < \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} >= \phi(v) \quad \forall v \in \mathcal{U}$$

where constraints $\phi(1) = 0$ on the load and $< u, 1 >= 0$ on the solution must be imposed in Problem C to make the problem uniquely solvable. Here $< \cdot, \cdot >$ denotes the L^2 inner product over the domain Ω.

Due to different boundary conditions in Problems A and C the asymptotic solution may fall into two different states when $\epsilon \to 0$. The first one occurring typically in conjunction with the Problem A has the asymptotic behavior

$$u \sim u_0 + \epsilon^2 u_1.$$

In this "cool" state no locking is present and the standard finite element scheme gives a nicely converging method as long as the solution u meets the usual regularity requirements.

On the other hand, in Problem C the solution u may behave as

$$u \sim \epsilon^{-2} u_0 + u_1$$

where u_0 satisfies the constraint

$$\frac{\partial u_0}{\partial \xi} = 0. \tag{5}$$

This "hot" state is prone to locking since due to the rotated coordinate system the bilinear finite element space is not capable of accommodating enough functions satisfying the constraint (5). This means that in Problem C the relative error in the energy norm satisfies only

$$\frac{|||u-u_h|||}{|||u|||} \sim \min\{1, \frac{h}{\epsilon}\}$$

even for infinitely smooth deformations u.

In order to avoid locking in the "hot" state a modified bilinear form can be introduced to reduce the constraint (5). In [I] and [IV] this form is chosen to be

$$\mathcal{A}_h(u,v) =< R_h\frac{\partial u}{\partial \xi}, R_h\frac{\partial v}{\partial \xi} > +\epsilon^2 < \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} > +\epsilon^2 < (I-R_h)\frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} > \quad (6)$$

where R_h is a numerical flux-reduction operator chosen to be the L^2-projection onto elementwise constant functions, i.e.

$$(R_h\varphi)(x,y) = \frac{1}{\text{area}(K)}\int_K \varphi(x',y')dx'dy' \quad (x,y)\in K$$

for every element K. With the use of (6) the constraint (5) is reduced to

$$R_h\frac{\partial u_0}{\partial \xi} = 0 \quad (7)$$

and the finite element approximation to u_0 is sought for in a larger space

$$\mathcal{U}_{0,h} = \{v \in \mathcal{U}_h \mid R_h\frac{\partial v}{\partial \xi} = 0\}.$$

Indeed, this approach is fruitful and it is shown in [I] that for sufficiently smooth solutions u the approximation error in Problem C in the "hot" state can be bounded as

$$e_A \leq Ch$$

on a uniform or piecewise uniform mesh. Also the consistency error stays bounded in both Problems, A and C. In particular it is shown that

$$e_C \leq Ch^2$$

in Problem A on a uniform or piecewise uniform mesh for a smooth u.

The above results can be refined to include several different types of meshes and more general boundary conditions. The analysis is based on noting that the condition (7) reduces to a difference scheme for a bilinear $v \in \mathcal{U}_h$ on a rectangular mesh. The question of the significance of the regularity of u is touched in [IV] where the asymptotic consistency error is discussed. Numerical experiments then show that for non-smooth u the performance of the reduced scheme can depend very subtly on the underlying mesh.

5 The shell problem

The two dimensional shell models used in finite element modeling of thin shells are derived from energy formulations of full 3D-models [15]. The shell model used in [II] and [III] is of Reissner-Naghdi type with two different scalings. In this model the scaled total of the shell is given by

$$\mathcal{F}_M(\underline{u}) = \frac{1}{2}\big(t^2\mathcal{A}_b(\underline{u},\underline{u}) + \mathcal{A}_m(\underline{u},\underline{u})\big) - Q(\underline{u}) \tag{8}$$

in the membrane-dominated case and by

$$\mathcal{F}_B(\underline{u}) = \frac{1}{2}\big(\mathcal{A}_b(\underline{u},\underline{u}) + t^{-2}\mathcal{A}_m(\underline{u},\underline{u})\big) - Q(\underline{u}) \tag{9}$$

in the bending-dominated case. Here $t \ll 1$ is the thickness of the shell, $\underline{u} = (u, v, w, \theta, \psi)$ is the vector of three translations and two rotations and Q is the load potential. The bilinear forms $\mathcal{A}_b(\underline{u},\underline{u})$ and $\mathcal{A}_m(\underline{u},\underline{u})$ represent the bending and membrane energies respectively and they are given by

$$\mathcal{A}_b(\underline{u},\underline{v}) = \int_\Omega \big\{\nu(\kappa_{11}+\kappa_{22})(\underline{u})(\kappa_{11}+\kappa_{22})(\underline{v}) + (1-\nu)\sum_{i,j=1}^{2}\kappa_{ij}(\underline{u})\kappa_{ij}(\underline{v})\big\}dxdy$$

and

$$\begin{aligned}\mathcal{A}_m(\underline{u},\underline{v}) = 6\gamma(1-\nu)&\int_\Omega\{\rho_1(\underline{u})\rho_1(\underline{v}) + \rho_2(\underline{u})\rho_2(\underline{v})\}dxdy \\ +12&\int_\Omega\{\nu(\beta_{11}+\beta_{22})(\underline{u})(\beta_{11}+\beta_{22})(\underline{v}) \\ &+ (1-\nu)\sum_{i,j=1}^{2}\beta_{ij}(\underline{u})\beta_{ij}(\underline{v})\}dxdy.\end{aligned}$$

where the integration is taken over the midsurface Ω of the shell. Further, ν is the Poisson ratio of the material, γ is a shear correction factor and κ_{ij}, β_{ij} and ρ_i represent the bending, membrane and transverse shear strains, respectively, depending on $\underline{u}$ as

$$\begin{aligned}\beta_{11} &= \frac{\partial u}{\partial x} + aw & \kappa_{11} &= \frac{\partial\theta}{\partial x} \\ \beta_{22} &= \frac{\partial v}{\partial y} + bw & \kappa_{22} &= \frac{\partial\psi}{\partial y} \\ \beta_{12} &= \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + cw = \beta_{21} & \kappa_{12} &= \frac{1}{2}(\frac{\partial\theta}{\partial y} + \frac{\partial\psi}{\partial x}) = \kappa_{21}\end{aligned}$$

and

$$\rho_1 = \theta - \frac{\partial w}{\partial x} \qquad \rho_2 = \psi - \frac{\partial w}{\partial y}.$$

The parameters a, b and c define the shell geometry, and the type of the shell is given by the discriminant $D = ab - c^2$. If $D > 0$ the shell is elliptic, if

$D < 0$ it is hyperbolic and the case $D = 0$ leads to a parabolic shell. The case $a = b = c = 0$ is excluded, since in this case the shell is totally flat, i.e. a plate.

The above model corresponds to the Reissner-Naghdi model of a shallow shell [18] where such a major simplifications are already made that up to the accuracy the geometry parameters can be taken constants to simplify the analysis. A relation of the above approach to the engineering practice in discussed in [7, 14].

The two different energy formulations (8), (9) lead to two different variational formulations:

(M) Find $\underline{u} \in \mathcal{U}_M$ such that

$$\mathcal{A}_M(\underline{u}, \underline{v}) = t^2\mathcal{A}_b(\underline{u}, \underline{v}) + \mathcal{A}_m(\underline{u}, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_M,$$

(B) Find $\underline{u} \in \mathcal{U}_B$ such that

$$\mathcal{A}_B(\underline{u}, \underline{v}) = \mathcal{A}_b(\underline{u}, \underline{v}) + t^{-2}\mathcal{A}_m(\underline{u}, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_B,$$

where $\mathcal{U}_M$ and $\mathcal{U}_B$ are the membrane and bending energy spaces, respectively. In a prototype setting the boundaries are assumed to free in the bending-dominated case (i.e. no constraints are imposed in $\mathcal{U}_B$, except those making the problem (B) uniquely solvable) and clamped in the membrane-dominated case (i.e. homogeneous boundary constraints are imposed in $\mathcal{U}_M$).

The problem of locking is associated with the bending-dominated scaling (B). Namely, if the set of inextensional deformations

$$\mathcal{U}_0 = \{\underline{v} \in \mathcal{U}_B \mid \mathcal{A}_m(\underline{v}, \underline{v}) = 0\}$$

has nonzero elements and $Q(\underline{v}) \neq 0$ for some $\underline{v} \in \mathcal{U}_0$ the standard finite element scheme suffers from locking. Also, letting $t \to 0$ in (B) leads to the inextensional formulation of the problem: Find $\underline{u} \in \mathcal{U}_0$ such that

$$\mathcal{A}_b(\underline{u}_0, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_0.$$

To avoid locking a modification of the membrane and transverse shear strains is needed. Several attempts in this direction mainly based on the mixed formulation of the problem include [1, 6, 8]. The difficulty does not lie in circumventing the locking effect, but in designing an element that has a good performance in both states, bending- and membrane-dominated. The following straightforward interpretation of the MITC4-element [14] is used in [II] and [III]. Let

$$\begin{aligned}\mathcal{A}_m^h(\underline{u}, \underline{v}) = 6\gamma(1-\nu) &\int_\Omega \{\tilde{\rho}_1(\underline{u})\tilde{\rho}_1(\underline{v}) + \tilde{\rho}_2(\underline{u})\tilde{\rho}_2(\underline{v})\}dxdy \\ &+12\int_\Omega \{\nu(\tilde{\beta}_{11} + \tilde{\beta}_{22})(\underline{u})(\tilde{\beta}_{11} + \tilde{\beta}_{22})(\underline{v}) \\ &\qquad + (1-\nu)\sum_{i,j=1}^{2} \tilde{\beta}_{ij}(\underline{u})\tilde{\beta}_{ij}(\underline{v})\}dxdy\end{aligned}$$

where

$$\tilde{\beta}_{11} = \Pi_h^x \beta_{11}, \quad \tilde{\beta}_{22} = \Pi_h^y \beta_{22}, \quad \tilde{\rho}_1 = \Pi_h^x \rho_1, \quad \tilde{\rho}_2 = \Pi_h^y \rho_2.$$

Here Π_h^x and Π_h^y are L^2-projections onto spaces $\mathcal{W}_h^x$ and $\mathcal{W}_h^y$ consisting of functions elementwise constant in x and linear in y or elementwise constant in y and linear in x. For the term β_{12} two different possibilities arise. The most natural one is to set $\tilde{\beta}_{12} = \Pi_h^{xy} \beta_{12}$ where $\Pi_h^{xy} = \Pi_h^x \Pi_h^y$ is the L^2-projection onto the space $\mathcal{W}_h^{xy}$ of elementwise constant functions. The other alternative is a bit more complicated in nature as shown in [14], but leads to similar results.

The above definitions lead to two different finite element schemes for finite element spaces $\mathcal{U}_{M,h} \subset \mathcal{U}_M$ and $\mathcal{U}_{B,h} \subset \mathcal{U}_B$:

(M_h) Find $\underline{u}_h \in \mathcal{U}_{M,h}$ such that

$$\mathcal{A}_M^h(\underline{u}_h, \underline{v}) = t^2 \mathcal{A}_b(\underline{u}_h, \underline{v}) + \mathcal{A}_m^h(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{M,h},$$

(B_h) Find $\underline{u}_h \in \mathcal{U}_{B,h}$ such that

$$\mathcal{A}_B^h(\underline{u}_h, \underline{v}) = \mathcal{A}_b(\underline{u}_h, \underline{v}) + t^{-2} \mathcal{A}_m^h(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{B,h},$$

together with the limit problem when $t \to 0$: Find $\underline{u}_h \in \mathcal{U}_{0,h}$ s.t.

$$\mathcal{A}_b(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{0,h}$$

where

$$\mathcal{U}_{0,h} = \{\underline{v} \in \mathcal{U}_{B,h} \mid \mathcal{A}_m^h(\underline{v}, \underline{v}) = 0\}.$$

The different variational formulations also give rise to different norms: $|||\cdot|||_{M,h} = \sqrt{\mathcal{A}_M^h(\cdot,\cdot)}$ and $|||\cdot|||_{B,h} = \sqrt{\mathcal{A}_B^h(\cdot,\cdot)} = t^{-2}|||\cdot|||_{M,h}$. The discretization error can again be split into the approximation error and the consistency error. In [II] and [III] two questions obtain the main attention:

1. In the bending-dominated case, how well the functions in $\mathcal{U}_0$ can be approximated by the functions in $\mathcal{U}_{0,h}$, i.e. given $\underline{u}_0 \in \mathcal{U}_0$ how large

$$e_a^0 = \inf_{\underline{v} \in \mathcal{U}_{0,h}} |||\underline{u}_0 - \underline{v}|||_{B,h}$$

is? Answer to this question reveals how well the formulation can avoid locking.

2. How large is the consistency error

$$e_{c,M} = \sup_{\underline{v} \in \mathcal{U}_{M,h}, \underline{v} \neq 0} \frac{(\mathcal{A}_M - \mathcal{A}_M^h)(\underline{u}, \underline{v})}{|||\underline{v}|||_{M,h}}$$

in the membrane-dominated case? Answer to this question gives information on how well the formulation can be used in the case of a membrane dominated deformation.

The performance of the modified finite element scheme depends on several properties of the domain Ω and the mesh. In this thesis the intention has been to find out the limits of the scheme under "optimal" conditions. Therefore the following assumptions are made:

1. The domain Ω is assumed to be of rectangular shape and periodic in one variable. This is required by the Fourier-methods used in the analysis of the deformation field $\underline{u}$.

2. The mesh is assumed to be rectangular and to have a constant mesh spacing in the periodic direction. This is necessary since the analysis relies on making a Discrete Fourier Transform of the functions in the finite element space.

Under these assumptions the main results in [II] and [III] state that

1. For $\underline{u}_0 \in \mathcal{U}_0$

$$e_a^0 \leq C_1 h |\underline{u}_0|_2 + C_2 h^{\frac{2}{3}(s-1)} |\underline{u}_0|_s, \quad 2 \leq s \leq 3$$

   where $C_2 = 0$ if the shell is elliptic, or if the mesh is aligned with the characteristics of the problem. Here $|\cdot|_k$ denotes the kth Sobolev-seminorm over Ω.

2. In the membrane-dominated case when $b \neq 0$ the consistency error satisfies

$$e_{c,M} \leq C_1(\underline{u}) h + C_2(t, s, \underline{u}) h^{1+s} + C_3(t, \underline{u}) h^2, \quad s \geq 0$$

   where

$$\begin{aligned} C_1(\underline{u}) =& C \sum_{ij} |\beta_{ij}(\underline{u})|_{2-m} \\ C_2(t, s, \underline{u}) =& C t^{-1} \sum_{ij} |\beta_{ij}(\underline{u})|_{1+s} \\ C_3(t, \underline{u}) =& C t^{-1} \sum_{i} |\rho_i(\underline{u})|_1. \end{aligned}$$

   Here $m = 0$ in the elliptic case and $m = 1$ in the parabolic and hyperbolic cases.

When interpreting the result for the consistency error $e_{c,M}$ it should be noted that the transverse shear strains ρ_i are typically very small at small t under smooth deformations so that the constant C_3 is not very likely to blow up when $t \to 0$.

Finally, in [V] a numerical performance of the modified scheme is confirmed using a well-known benchmark problem suffering from locking, the Morley shell [13].

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