

**THE PIECEWISE POLYNOMIAL COLLOCATION
METHOD FOR NONLINEAR WEAKLY
SINGULAR VOLTERRA EQUATIONS**

**Hermann Brunner
Arvet Pedas
Gennadi Vainikko**

Helsinki University of Technology
Institute of Mathematics
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Abstract. *Second-kind Volterra integral equations with weakly singular kernels typically have solutions which are nonsmooth near the initial point of the interval of integration. Using an adaptation of the analysis originally developed for nonlinear weakly singular Fredholm integral equations we present a complete discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for nonlinear Volterra integral equations with algebraic or logarithmic singularities in their kernels.*

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Key words. Nonlinear weakly singular Volterra and Fredholm integral equations, piecewise polynomial collocation, graded grids, optimal order of convergence

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¹Hermann Brunner

Department of Mathematics & Statistics

Memorial University of Newfoundland

St. John's, Nfld., Canada A1C 5S7

hbrunner@morgan.ucs.mun.ca

²Arvet Pedas

Department of Applied Mathematics

University of Tartu, Liivi 2-206

Tartu EE2400, Estonia

Arvet.Pedas@ut.ee

³Gennadi Vainikko

Institute of Mathematics

Helsinki University of Technology

P.O. Box 1100, FIN-02015 HUT, Finland

Gennadi.Vainikko@hut.fi

1. Introduction

The solution of a second kind Fredholm integral equation with weakly singular kernel is typically nonsmooth near the boundary of the domain of integration (its derivatives are unbounded). We refer to Richter (1976), Pedas (1977, 1979), Schneider (1979), Pitkäranta (1979, 1980), Vainikko and Pedas (1981), Graham (1982a), Vainikko, Pedas and Uba (1984), Vainikko (1991, 1993), Uba (1988), Kaneko, Noren and Xu (1990), Kangro R (1990), Kangro U (1990, 1993), Pedas and Vainikko (1994).

If one wants to obtain a high order convergence of a numerical method for these equations one has to take into account, in some way, the singular behaviour of the exact solution. It can be done using polynomial splines on special graded grids. The theory of graded grids in the approximation by polynomial splines goes back to Rice (1969) (for a complete theory see, for example, de Boor (1978) or Schumaker (1981)). In the numerical solution of second kind Fredholm integral equations with weakly singular kernels, graded grids were used by Chandler (1979), Schneider (1981), Vainikko and Uba (1981), Graham (1982b), Vainikko, Pedas and Uba (1984), Vainikko (1988, 1990, 1993), Hackbusch (1989), Uba (1989, 1994), Kaneko, Noren and Xu (1992), Tamme (1995), Pedas and Vainikko [23].

A similar situation arises in the case of Volterra integral equations. For second kind Volterra equations with weakly singular kernels the nonsmooth behaviour of solutions occurs near the initial point of integration. We refer to Miller and Feldstein (1971), de Hoog and Weiss (1974), Logan (1976), Lubich (1983), Brunner (1983, 1985b), Brunner and van der Houwen (1986). Again, if one is interested in finding an approximate solution which exhibits high order accuracy, then one may resort to approximation with polynomial splines on graded grids which reflect the singular behaviour of the exact solution near the initial point. In the numerical solution of second kind Volterra integral equations with weakly singular kernels graded grids were used by Brunner (1984, 1985a), Brunner and van der Houwen (1986), and Brunner (1987).

In the previous works the case of Fredholm equations was considered independently of Volterra equations and vice versa, the results about Volterra equations were obtained independently of existing results about Fredholm equations.

The purpose of the present paper is to show how in the numerical solution of weakly singular Volterra integral equations by polynomial splines on graded grids it is possible to use corresponding existing results about Fredholm equations. More precisely, on the basis of results from [39, 22-23] we shall study the smoothness of the solution and the piecewise polynomial collocation method for a sufficiently wide class of nonlinear weakly singular Volterra integral equations. Using special collocation points with graded grids we derive global convergence estimates and analyze a superconvergence effect at collocation points. The main results of the paper considerably extend known ones and are formulated in Theorems 2.1–2.4 (see Section 2). The proofs of these assertions are given in Section 4 and are based on the following simple idea: for a given Volterra equation

$$(1.1) \quad u(t) = \int_0^t K(t, s, u(s)) ds + f(t), \quad 0 \leq t \leq T,$$

we find the appropriate extensions $\tilde{K}(t, s, u)$ and $\tilde{f}(t)$ for $K(t, s, u)$ and $f(t)$ so that from

the corresponding results of Section 3 about the Fredholm equation

$$u(t) = \int_0^{2T} \tilde{K}(t, s, u(s)) ds + \tilde{f}(t), \quad 0 \leq t \leq 2T,$$

we can derive all results formulated in Theorems 2.1–2.4 about Volterra equation (1.1). Finally, in Section 5 we present some numerical illustrations.

2. Volterra equations with weakly singular kernels

2.1. Smoothness of the solution. Consider the nonlinear Volterra equation

$$(2.1) \quad u(t) = \int_0^t K(t, s, u(s)) ds + f(t), \quad 0 \leq t \leq T.$$

The following assumptions (V1) – (V3) are made.

(V1) The kernel $K = K(t, s, u)$ is m times ($m \geq 1$) continuously differentiable with respect to t, s, u for $t \in [0, T], s \in [0, t], u \in \mathbb{R}$, and there exists a real number $\nu \in (-\infty, 1)$ such that for $0 \leq s < t \leq T, u \in \mathbb{R}$, and for nonnegative integers i, j, k with $i + j + k \leq m$, the following inequalities hold:

$$(2.2) \quad \left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u) \right| \leq b_1(|u|) \begin{cases} 1 & \text{if } \nu + i < 0 \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0 \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0 \end{cases}$$

and

$$(2.3) \quad \left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u_1) - \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^k K(t, s, u_2) \right| \\ \leq b_2(\max\{|u_1|, |u_2|\}) |u_1 - u_2| \begin{cases} 1 & \text{if } \nu + i < 0 \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0 \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0 \end{cases}$$

where the functions $b_1: [0, \infty) \rightarrow [0, \infty)$ and $b_2: [0, \infty) \rightarrow [0, \infty)$ are assumed to be monotonically increasing.

(V2) $f \in C^{m, \nu}(0, T]$, i.e. $f(t)$ is m times continuously differentiable for $0 < t \leq T$ and the estimate

$$(2.4) \quad \left| f^{(k)}(t) \right| \leq \text{const}_f \begin{cases} 1 & \text{if } k < 1 - \nu \\ 1 + |\log t| & \text{if } k = 1 - \nu \\ t^{1 - \nu - k} & \text{if } k > 1 - \nu \end{cases}, \quad t \in (0, T],$$

holds for $k = 0, 1, \dots, m$.

(V3) The integral equation (1.1) has a solution $u_0 \in L^\infty(0, T)$.

Notice that conditions (V1) and (V2) guarantee the existence and uniqueness of the solution to (2.1) on some interval $[0, T_0]$, $T_0 \leq T$. On $[0, T]$ the existence and uniqueness of the solution will be guaranteed if we impose the following global Lipschitz condition on $K(t, s, u)$: for $0 \leq t \leq T$, $0 \leq s < t$, $u_1, u_2 \in \mathbb{R}$, there holds

$$|K(t, s, u_1) - K(t, s, u_2)| \leq b|u_1 - u_2| \begin{cases} 1 & \text{if } \nu < 0 \\ 1 + |\log|t - s|| & \text{if } \nu = 0 \\ |t - s|^{-\nu} & \text{if } \nu > 0 \end{cases}$$

with a constant b independent of u_1 and u_2 (cf. (2.3), $i = j = k = 0$). We do not restrict the problem by this global Lipschitz condition but assume (V3). Moreover assuming (V3), we actually could replace (V1) by the corresponding local condition for $t \in [0, T]$, $s \in [0, t]$, $|u - u_0(s)| \leq \delta$ with a $\delta > 0$. Outside this set, the kernel $K(t, s, u)$ is involved neither in (2.1) nor in the collocation method for it described in Section 2.2. Nevertheless, we remain with the formulation of (V1) given above.

It follows from (V1) that the kernel $K(t, s, u)$ may have a weak singularity as $s \rightarrow t$ ($i = j = k = 0$, $0 \leq \nu < 1$). In the case $\nu < 0$, the kernel $K(t, s, u)$ is bounded for $0 \leq s \leq t \leq T$ and fixed $u \in \mathbb{R}$, but its derivatives may be singular as $s \rightarrow t$. Often the kernel K has the form

$$K(t, s, u) = a(t, s, u)(t - s)^{-\beta}, \quad \beta < 1,$$

or

$$K(t, s, u) = a(t, s, u) \log(t - s)$$

where $a(t, s, u)$ is an m -smooth function of its arguments for $0 \leq s \leq t \leq T$, $u \in \mathbb{R}$. Clearly, condition (V1) is satisfied in these examples with $\nu = \beta$ and $\nu = 0$, respectively. Actually, (V1) remains fulfilled even if derivatives of $a(t, s, u)$ have some (sufficiently weak) singularities as $s \rightarrow t$.

The following result states the regularity properties of solutions of equation (2.1).

Theorem 2.1. Let the conditions (V1) and (V2) be fulfilled. If equation (2.1) has a solution u in $L^\infty(0, T)$, then $u \in C^{m, \nu}(0, T]$.

The proof of Theorem 2.1 is given in Section 4. We remark that smoothness properties of solutions to (more special) weakly singular Volterra equations are analyzed in [19, 6].

Notice also that $u \in C^{m, \nu}(0, T]$, having an integrable derivative in $(0, T]$ can be extended up to a continuous function on $[0, T]$. The extended function will be denoted again by u .

2.2. Piecewise polynomial approximation of the solution. For given $N \in \mathbb{N}$ let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the interval $[0, T]$ with grid points

$$(2.5) \quad t_j \equiv t_j^{(N)} = (j/N)^r T, \quad j = 0, 1, \dots, N,$$

where $r \in \mathbb{R}$, $r \geq 1$. For $r = 1$, the grid points t_0, t_1, \dots, t_N are uniformly located; for $r > 1$, the grid points are more densely located towards the left endpoint of the interval $[0, T]$. In every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) we choose m collocation points

$$(2.6) \quad \xi_{ji} = t_j + \frac{\eta_i + 1}{2}(t_{j+1} - t_j), \quad i = 1, \dots, m,$$

where η_1, \dots, η_m do not depend on N ,

$$(2.7) \quad -1 \leq \eta_1 < \eta_2 < \dots < \eta_m \leq 1.$$

To a continuous function $u: [0, T] \rightarrow \mathbb{R}$ we assign a piecewise polynomial interpolant $P_N u: [0, T] \rightarrow \mathbb{R}$ as follows: 1) on every subinterval $[t_{j-1}, t_j]$, ($j = 1, \dots, N$) $P_N u$ is a polynomial of degree $m - 1$; 2) $P_N u$ interpolates u at points $\xi_{j1}, \dots, \xi_{jm}$:

$$(P_N u)(\xi_{ji}) = u(\xi_{ji}), \quad i = 1, \dots, m; \quad j = 1, \dots, N.$$

Thus, the interpolation function $(P_N u)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) and may be discontinuous at the interior grid points $t = t_j$ ($j = 1, \dots, N - 1$). We may treat $P_N u$ as a two-valued function in these points. Note that in the case $\eta_1 = -1$ and $\eta_m = 1$, $P_N u$ is a continuous function on $[0, T]$.

We denote by E_N the range of the interpolatory projection P_N , i.e. the set of all piecewise polynomial functions on $[0, T]$ which are polynomials of degree not exceeding $m - 1$ on every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$). We introduce also the notation

$$h = T/N.$$

We look for an approximate solution $u_N \in E_N$ to the integral equation (2.1) which satisfies this equation at the collocation points (2.6):

$$(2.8) \quad [u_N(t) - \int_0^t K(t, s, u_N(s))ds - f(t)]_{t=\xi_{ji}} = 0, \quad i = 1, \dots, m; \quad j = 1, \dots, N.$$

Conditions (2.8) form a system of equations whose exact form is determined by a basis of E_N . For instance, in the interval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) we may use the representation

$$(2.9) \quad u_N(s) = \sum_{i=1}^m c_{ji} \varphi_{ji}(s), \quad s \in [t_{j-1}, t_j],$$

where $\varphi_{ji}(s)$ ($s \in [t_{j-1}, t_j]$) is the polynomial of degree $m - 1$ such that

$$\varphi_{ji}(\xi_{jk}) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}, \quad k = 1, \dots, m.$$

The collocation conditions (2.8) take the form of the (nonlinear) system of algebraic equations to determine the coefficients $\{c_{ji}\}$:

$$(2.10) \quad c_{ji} = f(\xi_{ji}) + \sum_{l=1}^{j-1} \int_{t_{l-1}}^{t_l} K(\xi_{ji}, s, \sum_{k=1}^m c_{lk} \varphi_{lk}(s)) ds + \\ + \int_{t_{j-1}}^{\xi_{ji}} K(\xi_{ji}, s, \sum_{k=1}^m c_{jk} \varphi_{jk}(s)) ds, \quad i = 1, \dots, m; j = 1, \dots, N.$$

The coefficients c_{11}, \dots, c_{1m} can be found from the system

$$(2.11) \quad c_{1i} = f(\xi_{1i}) + \int_0^{\xi_{1i}} K(\xi_{1i}, s, \sum_{k=1}^m c_{1k} \varphi_{1k}(s)) ds, \quad i = 1, \dots, m.$$

Using c_{11}, \dots, c_{1m} one can find c_{21}, \dots, c_{2m} from the system

$$c_{2i} = f(\xi_{2i}) + \int_0^{t_1} K(\xi_{2i}, s, \sum_{k=1}^m c_{1k} \varphi_{1k}(s)) ds + \\ + \int_{t_1}^{\xi_{2i}} K(\xi_{2i}, s, \sum_{k=1}^m c_{2k} \varphi_{2k}(s)) ds, \quad i = 1, \dots, m.$$

Generally, using $c_{11}, \dots, c_{1m}, \dots, c_{j-1,1}, \dots, c_{j-1,m}$ the coefficients c_{j1}, \dots, c_{jm} can be found from the m equations (2.10) with corresponding j . Thus at every step (on every subinterval $[t_{j-1}, t_j]$) one has to solve a system of m nonlinear equations; the initial guess can be chosen using the solution on the previous subinterval. For the first block system (2.11) a suitable initial guess is $c_{1i} = f(\xi_{1i})$, $i = 1, \dots, m$. The following theorem states the global convergence rate for the collocation method (2.8).

Theorem 2.2 Let assumptions (V1)-(V3) be fulfilled and let the collocation points (2.6) with gridpoints (2.5) be used.

Then there exist an $N_0 \in \mathbb{N}$ and a real number $\delta_0 > 0$ such that, for $N > N_0$, the collocation method (2.8) defines a unique approximation $u_N \in E_N$ to the solution u_0 of equation (2.1) satisfying $\|u_N - u_0\|_{L^\infty(0,T)} \leq \delta_0$. The following error estimates hold:

1) if $m < 1 - \nu$ then

$$(2.12) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq ch^m \quad \text{for } r \geq 1;$$

2) if $m = 1 - \nu$ then

$$(2.13) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^m(1 + |\log h|) & \text{for } r = 1 \\ h^m & \text{for } r > 1 \end{cases}$$

and

$$(2.14) \quad \|u_N - u_0\|_{L^p(0,T)} \leq ch^m \quad \text{for } r \geq 1, 1 \leq p < \infty;$$

3) if $m > 1 - \nu$ then

$$(2.15) \quad \max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^{r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu} \\ h^m & \text{for } r \geq \frac{m}{1-\nu} \end{cases}$$

and for $1 \leq p < \infty$,

$$(2.16) \quad \|u_N - u_0\|_{L^p(0,T)} \leq c \begin{cases} h^{r(1-\nu+\frac{1}{p})} & \text{if } 1 \leq r < \frac{m}{1-\nu+\frac{1}{p}}, m > 1 - \nu + \frac{1}{p} \\ h^m(1 + |\log h|)^{\frac{1}{p}} & \text{if } r = \frac{m}{1-\nu+\frac{1}{p}}, m \geq 1 - \nu + \frac{1}{p} \\ h^m & \text{if } r > \frac{m}{1-\nu+\frac{1}{p}}, r \geq 1 \end{cases};$$

4) if $r = r(m, \nu) \geq 1$ is restricted by conditions

$$(2.17) \quad \left\{ \begin{array}{ll} r > \frac{m}{2(1-\nu)} & \text{for } 0 \leq \nu < 1 \\ r > \frac{m}{2-\nu} & \text{for } \nu < 0 \\ r \geq 1 & \text{for } \nu < 0, \nu < -(m-2) \end{array} \right\}$$

then

$$(2.18) \quad \varepsilon_N \leq ch^m,$$

where

$$(2.19) \quad \varepsilon_N = \max_{i=1, \dots, m; j=1, \dots, N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|$$

is the maximal error of the approximate solution $u_N \in E_N$ at the collocation points (2.6). The constants c in (2.12)-(2.18) are independent of $h = T/N$.

The proof of Theorem 2.2 is given in Section 4.

2.3. Superconvergence at collocation points. Now we assume that the points η_1, \dots, η_m in (2.6) (the points (2.7)) are the knots of a quadrature formula

$$(2.20) \quad \int_{-1}^1 \varphi(\xi) d\xi \approx \sum_{i=1}^m w_i \varphi(\eta_i), \quad -1 \leq \eta_1 < \dots < \eta_m \leq 1,$$

which is exact for polynomials of degree $m + \mu$, $\mu \in \mathbb{Z}$, $0 \leq \mu \leq m - 1$. Actually, the weights w_1, \dots, w_m of the quadrature formula (2.20) will not be used in our algorithms.

The case $\mu = m - 1$ corresponds to the Gauss quadrature formula and is of the greatest interest in the following theorem.

Theorem 2.3. Let the following conditions be fulfilled:

(V1') The kernel $K(t, s, u)$ and $\partial K(t, s, u)/\partial u$ are $m + \mu + 1$ times ($m, \mu \in \mathbb{Z}$, $m \geq 1$, $0 \leq \mu \leq m - 1$) continuously differentiable with respect to t, s, u for $t \in [0, T]$, $s \in [0, t]$, $u \in \mathbb{R}$, and satisfy (2.2) and (2.3) with $i + j + k \leq m + \mu + 1$, $\nu \in (-\infty, 1)$.

(V2') $f \in C^{m+\mu+1, \nu}(0, T]$.

(V3) The integral equation (2.1) has a solution $u_0 \in L^\infty(0, T)$.

(V4) The collocation points (2.6) are generated by the knots (2.7) of a quadrature formula (2.20) which is exact for all polynomials of degree $m + \mu$, $0 \leq \mu \leq m - 1$.

(V5) The scaling parameter $r = r(m, \nu, \mu) \geq 1$ is subject to the restrictions

$$(2.21) \quad \left\{ \begin{array}{ll} r > \frac{m}{1-\nu}, r \geq \frac{m+1-\nu}{2-\nu} & \text{if } 1-\nu < \mu+1 \\ r > \frac{m}{1-\nu}, r > \frac{m+\mu+1}{2-\nu} & \text{if } \mu+1 \leq 1-\nu < m \\ r \geq \frac{m+\mu+1}{2-\nu}, r > 1 & \text{if } 1-\nu = m \\ r \geq \frac{m+\mu+1}{2-\nu} & \text{if } 1-\nu > m \end{array} \right\}.$$

Then

$$(2.22) \quad \varepsilon_N \leq ch^m \begin{cases} h & \text{if } \nu < 0 \\ h(1 + |\log h|) & \text{if } \nu = 0 \\ h^{1-\nu} & \text{if } \nu > 0 \end{cases}$$

where ε_N is defined in (2.19) and c is a positive constant which is independent of $h = T/N$.

The proof of Theorem 2.3 is given in Section 4.

Theorem 2.4. Let the conditions of Theorem 2.3 be fulfilled. Assume additionally that:

(V6) $\nu < 0$, $\mu \geq 1$, and for $0 \leq j \leq \min\{\mu - 1, -\nu\}$, $0 \leq k \leq \min\{\mu - 1, -\nu\}$, the derivatives

$$(2.23) \quad \left(\frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^{k+1} K(t, s, u)$$

are bounded and continuous for $0 \leq t \leq T$, $0 \leq s \leq t$, $|u| \leq \tau$ with any $\tau > 0$, and

$\left(\frac{\partial}{\partial s} \right)^j \left(\frac{\partial}{\partial u} \right)^{k+1} K(t, s, u) \rightarrow 0$ as $s \rightarrow t - 0$.

Then

$$(2.24) \quad \varepsilon_N \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1-\nu > \mu+1 \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1-\nu = \mu+1 \\ h^{1-\nu} & \text{if } 1-\nu < \mu+1 \end{cases}$$

where ε_N is defined in (2.19) and the constant c is independent of $h = T/N$.

The proof of Theorem 2.4 is given in Section 4.

Remark 2.1 (comment on the additional condition (V6) of Theorem 2.4). Assumption (V1') of Theorem 2.3 guarantees the boundedness and continuity of the derivatives (2.23) for $j < \min\{\mu - 1, -\nu\}$, $j + k \leq m + \mu + 1$, on any set $0 \leq s < t \leq T$, $-\tau < u < \tau$, $\tau > 0$; for $j = -\nu$ with $\nu \in \mathbb{Z}$, $-\nu < \mu - 1$, a logarithmical singularity may occur. Condition (V6) bans this possible singularity.

Remark 2.2. An estimation $\max_{0 \leq t \leq T} |u_N(t) - u_0(t)| \leq ch^m$ is of optimal order even for a function $u \in C^\infty[0, T]$. Theorem 2.2 shows that, using sufficiently great values of the scaling parameters r , the optimal accuracy $O(h^m)$ can be achieved for collocation method (2.8). Theorems 2.3 and 2.4 show that the superconvergence phenomenon at collocation points takes place.

Remark 2.3. Under the conditions of Theorem 2.3

$$\max_{0 \leq t \leq T} |\tilde{u}_N(t) - u_0(t)| \leq ch^m \begin{cases} h & \text{if } \nu < 0 \\ h(1 + |\log h|) & \text{if } \nu = 0 \\ h^{1-\nu} & \text{if } \nu > 0 \end{cases}$$

where

$$\tilde{u}_N(t) = \int_0^t K(t, s, u_N(s)) ds + f(t).$$

Under the conditions of Theorem 2.4,

$$\max_{0 \leq t \leq T} |\tilde{u}_N(t) - u_0(t)| \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1 - \nu > \mu + 1 \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1 - \nu = \mu + 1 \\ h^{1-\nu} & \text{if } 1 - \nu < \mu + 1 \end{cases}.$$

3. Fredholm equations with weakly singular kernels

In this Section we formulate some results from [39,22-23] about Fredholm integral equations which give a basis for the proofs of the results formulated in Section 2.

Consider a nonlinear Fredholm integral equation

$$(3.1) \quad u(t) = \int_0^{2T} K(t, s, u(s)) ds + f(t), \quad 0 \leq t \leq 2T.$$

The following assumptions (F1) and (F2) are made.

(F1) The kernel $K = K(t, s, u)$ is m times ($m \geq 1$) continuously differentiable with respect to t, s, u for $t, s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, whereby there exists a real number $\nu \in (-\infty, 1)$ such that for nonnegative integers i, j, k with $i + j + k \leq m$ inequalities (2.2) and (2.3) hold.

(F2) $f \in C^{m, \nu}(0, 2T)$, i.e. $f(t)$ is m times continuously differentiable for $0 < t < 2T$, and the estimate

$$(3.2) \quad \left| f^{(k)}(t) \right| \leq \text{const}_f \begin{cases} 1 & \text{if } k < 1 - \nu \\ 1 + |\log \varrho(t)| & \text{if } k = 1 - \nu \\ \varrho(t)^{1-\nu-k} & \text{if } k > 1 - \nu \end{cases}, \quad t \in (0, 2T),$$

holds for $k = 0, 1, \dots, m$, where $\varrho(t) = \min\{t, 2T - t\}$.

Theorem 3.1 [22, 39, p.137]. Let conditions (F1) and (F2) be fulfilled. If the integral equation (3.1) has a solution $u \in L^\infty(0, 2T)$, then $u \in C^{m,\nu}(0, 2T)$.

In the sequel we shall assume that:

(F3) The integral equation (3.1) has a solution $u_0 \in L^\infty(0, 2T)$ and the linearized equation

$$(3.3) \quad v(t) = \int_0^{2T} K_0(t, s)v(s)ds, \quad K_0(t, s) = \partial K(t, s, u)/\partial u \Big|_{u=u_0(s)},$$

has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$.

For given $N \in \mathbb{N}$, let $0 = t_0 < t_1 < \dots < t_{2N} = 2T$ be a partition of the interval $[0, 2T]$ with gridpoints (2.5) and gridpoints t_{N+1}, \dots, t_{2N} which are got by reflecting t_{N-1}, \dots, t_0 with respect to $t_N = T$:

$$(3.4) \quad t_j = (j/N)^r T, \quad j = 0, 1, \dots, N; \quad t_{j+N} = 2T - t_{N-j}, \quad j = 1, \dots, N.$$

In every subinterval $[t_{j-1}, t_j]$ ($j = 1, \dots, 2N$) we choose m collocation points (2.6) determined by points (2.7). We denote by \tilde{E}_N the set of all piecewise polynomial functions on $[0, 2T]$ which are polynomials of degree not exceeding $m - 1$ on every interval $[t_{j-1}, t_j]$ ($j = 1, \dots, 2N$). We look for an approximation solution $u_N \in \tilde{E}_N$ to the integral equation (3.1) which satisfies this equation at the collocation points (2.6) corresponding to the gridpoints (3.4):

$$(3.5) \quad \left[u_N(t) - \int_0^{2T} K(t, s, u_N(s))ds - f(t) \right]_{t=\xi_{ji}} = 0,$$

$$i = 1, \dots, m; \quad j = 1, \dots, 2N.$$

Theorem 3.2 (see [39, p.143], cf. also [42, 41]). Let the conditions (F1)–(F3) be fulfilled and let the collocation points (2.6) with gridpoints (3.4) be used.

Then there exist an $N_0 \in \mathbb{N}$ and a real number $\delta_0 > 0$ such that for $N > N_0$, the collocation method (3.5) defines a unique approximation $u_N \in \tilde{E}_N$ to the solution u_0 of equation (3.1) satisfying $\|u_N - u_0\|_{L^\infty(0,2T)} \leq \delta_0$. The following error estimates hold:

1) if $m < 1 - \nu$ then

$$(3.6) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq ch^m \quad \text{for } r \geq 1;$$

2) if $m = 1 - \nu$ then

$$(3.7) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^m(1 + \log h) & \text{for } r = 1 \\ h^m & \text{for } r > 1 \end{cases}$$

and

$$(3.8) \quad \|u_N - u_0\|_{L^p(0,2T)} \leq ch^m \quad \text{for } r \geq 1, \quad 1 \leq p < \infty;$$

3) if $m > 1 - \nu$ then

$$(3.9) \quad \max_{0 \leq t \leq 2T} |u_N(t) - u_0(t)| \leq c \begin{cases} h^{r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu} \\ h^m & \text{for } r \geq \frac{m}{1-\nu} \end{cases},$$

and

$$(3.10) \quad \|u_N - u_0\|_{L^p(0,2T)} \leq c \begin{cases} h^{r(1-\nu+\frac{1}{p})} & \text{if } 1 \leq r < \frac{m}{1-\nu+\frac{1}{p}}, m > 1 - \nu + \frac{1}{p} \\ h^m(1 + |\log h|)^{\frac{1}{p}} & \text{if } r = \frac{m}{1-\nu+\frac{1}{p}}, m \geq 1 - \nu + \frac{1}{p} \\ h^m & \text{if } r > \frac{m}{1-\nu+\frac{1}{p}}, r \geq 1 \end{cases},$$

for $1 \leq p < \infty$;

4) if $r = r(m, \nu) \geq 1$ satisfies (2.4) then

$$(3.11) \quad \tilde{\varepsilon}_N \leq ch^m$$

where

$$(3.12) \quad \tilde{\varepsilon}_N = \max_{i=1, \dots, m; j=1, \dots, 2N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|.$$

Theorem 3.3. [23] Let the following conditions be fulfilled:

(F1') The kernel $K(t, s, u)$ and $\partial K(t, s, u)/\partial u$ are $m + \mu + 1$ times ($m, \mu \in \mathbb{Z}$, $m \geq 1$, $0 \leq \mu \leq m - 1$) continuously differentiable with respect to t, s, u for $t, s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, and satisfy (2.2) and (2.3) with $i + j + k \leq m + \mu + 1$, $-\infty < \nu < 1$.

(F2') $f \in C^{m+\mu+1, \nu}(0, 2T)$.

(F3) The integral equation (3.1) has a solution $u_0 \in L^\infty(0, 2T)$, and the linearized equation (3.3) has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$.

(F4) The collocation points (2.6) are generated by the knots (2.7) of a quadrature formula (2.20) which is exact for all polynomials of degree $m + \mu$, $0 \leq \mu \leq m - 1$.

(F5) The scaling parameter $r = r(m, \nu, \mu) \geq 1$ satisfies the conditions (2.18).

Then

$$\tilde{\varepsilon}_N \leq ch^m \begin{cases} h & \text{if } \nu < 0 \\ h(1 + |\log h|) & \text{if } \nu = 0, \\ h^{1-\nu} & \text{if } \nu > 0 \end{cases},$$

where $\tilde{\varepsilon}_N$ is defined in (3.12).

In addition we assume now that:

(F6) $\nu < 0$, $\mu \geq 1$, and for $0 \leq j \leq \min\{\mu - 1, -\nu\}$, $0 \leq k \leq \min\{\mu - 1, -\nu\}$, the derivatives (2.23) are bounded and continuous on $[0, 2T] \times [0, 2T] \times [-\tau, \tau]$ with any $\tau > 0$.

Then

$$\tilde{\varepsilon}_N \leq ch^m \begin{cases} h^{\mu+1} & \text{if } 1 - \nu > \mu + 1 \\ h^{\mu+1}(1 + |\log h|) & \text{if } 1 - \nu = \mu + 1. \\ h^{1-\nu} & \text{if } 1 - \nu < \mu + 1 \end{cases}.$$

4. Proof of Theorems 2.1–2.4

Assume that the kernel $K(t, s, u)$ of equation (2.1) satisfies (V1) for $0 \leq s < t \leq T$, $-\infty < u < \infty$. Assume also that the forcing function $f(t)$ of equation (2.1) satisfies (V2) for $0 < t \leq T$, and let $u_0 \in L^\infty(0, T)$ be a solution of equation (2.1). First of all we extend $K(t, s, u)$ up to a $\tilde{K}(t, s, u)$ which will satisfy condition (F1) for $0 \leq t \leq 2T$, $0 \leq s \leq 2T$, $t \neq s, u \in \mathbb{R}$.

We shall use an extension method by reflecting (see, for example [17]): for a given function $v \in C^m[a - \tau, a]$, $m \in \mathbb{N}$, $a, \tau \in \mathbb{R}$, $\tau > 0$, put

$$(4.1) \quad \tilde{v}(s) = \begin{cases} v(s) & \text{for } s \in [a - \tau, a] \\ \sum_{j=0}^m c_j v(a - j(s - a)) & \text{for } s \in (a, a + \frac{\tau}{m}] \end{cases}$$

where

$$(4.2) \quad \sum_{j=0}^m (-j)^k c_j = 1, \quad k = 0, 1, \dots, m.$$

From (4.1) and (4.2) one obtains that

$$\tilde{v} \in C^m \left[a - \tau, a + \frac{\tau}{m} \right].$$

Now we construct an extension for the kernel $K(t, s, u)$ of equation (1.1). First, for any $u \in \mathbb{R}$, $\frac{2}{3}T \leq t \leq T$, we extend $K(t, s, u)$ with respect to s for $s < 0$ (denoting this extension again by $K(t, s, u)$):

$$K(t, s, u) = e(-s) \sum_{j=0}^m c_j K(t, -js, u), \quad -\frac{2T}{3m} \leq s < 0.$$

Here (c_0, c_1, \dots, c_m) is the (unique) solution of the system (4.2) and $e \in C^\infty(\mathbb{R}_+)$ is a smooth function such that

$$(4.3) \quad e(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \frac{2T}{9m} \\ 0 & \text{for } \frac{2T}{6m} \leq s \leq T \end{cases}.$$

After that, for any $u \in \mathbb{R}$, we extend $K(t, s, u)$ with respect to t ($T \leq t \leq 2T$) along the lines $s = t - \gamma$ ($0 < \gamma \leq 2T$):

$$K_1(t, s, u) = \begin{cases} K(t, s, u) & \text{for } 0 \leq s < t \leq T \\ e(t - T) \sum_{j=0}^m c_j K(T - j(t - T), T - j(s + \gamma - T) - \gamma, u) & \text{for } T \leq s < t \leq 2T, t - s = \gamma. \end{cases}$$

Finally, in the third step we put

$$(4.4) \quad \tilde{K}(t, s, u) = \begin{cases} K_1(t, s, u)e_1(u) & \text{for } 0 \leq s < t \leq 2T \\ 0 & \text{for } 0 \leq t < s \leq 2T \end{cases}$$

with a $e_1 \in C_0^\infty(\mathbb{R})$, $e_1(u) = 1$ for $|u| \leq 2M$, $e_1(u) = 0$ for $|u| \geq 3M$, where $M = \max_{0 \leq t \leq T} |u_0(t)|$. It is easy to check that the function $\tilde{K}(t, s, u)$ satisfies (F1).

Indeed, using (V1) we obtain that $\tilde{K}(t, s, u)$ is m times continuously differentiable with respect to t, s, u for $t \in [0, 2T]$, $s \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$. Further, it follows from the above construction that:

1) if $0 \leq s < t \leq T$, $u \in \mathbb{R}$, then

$$\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) = \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k [K(t, s, u)e_1(u)];$$

2) if $T \leq s < t \leq 2T$, $u \in \mathbb{R}$, then

$$(4.5) \quad \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) =$$

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k e(t-T) \right] \sum_{p=0}^m c_p K(T-p(t), T-p(s+\gamma-T)-\gamma, u) e_1(u) + \\ & + e(t-T) \sum_{p=0}^m c_p \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \times \\ & \times [K(T-p(t-T), T-p(s+\gamma-T)-\gamma, u) e_1(u)]; \end{aligned}$$

3) if $0 \leq t < s \leq 2T$ then

$$\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) = 0.$$

In case 1) we obtain estimates (2.2) and (2.3) for $\tilde{K}(t, s, u)$ using Assumption (V1). In case 3) statements (2.2) and (2.3) with respect to $\tilde{K}(t, s, u)$ are trivially fulfilled. Consider the case 2). Using (4.3), (4.5) and (V1) we obtain

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \left(\frac{\partial}{\partial u}\right)^k \tilde{K}(t, s, u) \right| \leq$$

$$\begin{aligned} &\leq cb_1(3M) \left[\left\{ \begin{array}{ll} 1 & \text{if } \nu < 0 \\ 1 + |\log |t - s|| & \text{if } \nu = 0 \\ |t - s|^{-\nu} & \text{if } \nu > 0 \end{array} \right\} + \left\{ \begin{array}{ll} 1 & \text{if } \nu + i < 0 \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0 \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0 \end{array} \right\} \right] \leq \\ &\leq c'b_1(3M) \left\{ \begin{array}{ll} 1 & \text{if } \nu + i < 0 \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0 \\ |t - s|^{-\nu - i} & \text{if } \nu + i > 0 \end{array} \right\}. \end{aligned}$$

In a similar way we obtain that condition (2.3) is also satisfied.

So, we have shown that the function $\tilde{K}(t, s, u)$ is m times continuously differentiable with respect to t, s, u for $t, s, u \in [0, 2T]$, $t \neq s$, $u \in \mathbb{R}$, and satisfies conditions (2.2) and (2.3) with some b_1 and b_2 independent of $|u|$, respectively, $\max\{|u_1|, |u_2|\}$. Thus, $\tilde{K}(t, s, u)$ satisfies also the global Lipschitz condition: for $t \in [0, 2T]$, $s \in [0, t)$, $u_1, u_2 \in \mathbb{R}$,

$$(4.6) \quad |\tilde{K}(t, s, u_1) - \tilde{K}(t, s, u_2)| \leq \tilde{b} \left\{ \begin{array}{ll} 1 & \text{if } \nu < 0 \\ 1 + |\log |t - s|| & \text{if } \nu = 0 \\ |t - s|^{-\nu} & \text{if } \nu > 0 \end{array} \right\}$$

with a constant \tilde{b} independent of u_1 and u_2 .

Further we put

$$(4.7) \quad \tilde{f}(t) = \begin{cases} f(t) & \text{for } 0 < t \leq T \\ e(t - T) \sum_{j=0}^m c_j f(T - j(t - T)) & \text{for } T < t \leq 2T \end{cases}$$

where (c_0, c_1, \dots, c_m) is the solution of the system (4.2) and $e \in C^\infty(\mathbb{R}_+)$ is defined in (4.3). It is clear that $\tilde{f} \in C^{m, \nu}(0, 2T) \subset C[0, 2T]$.

Consider the equation

$$(4.8) \quad u(t) = \int_0^{2T} \tilde{K}(t, s, u(s)) ds + \tilde{f}(t), \quad 0 \leq t \leq 2T,$$

where \tilde{K} and \tilde{f} are defined by (4.4) and (4.7). Equation (4.8) is actually a Volterra integral equation, since $\tilde{K}(t, s, u) = 0$ for $0 \leq t < s \leq 2T$. Due to the weak singularity of the kernel $\tilde{K}(t, s, u)$ and the global Lipschitz condition (4.6), (4.8) is uniquely solvable in $C[0, 2T]$; let $\tilde{u}_0 \in C[0, 2T]$ denote the solution. It follows from the above construction that $\tilde{u}_0(t) = u_0(t)$ for $0 \leq t \leq T$. According to Theorem 3.1, $\tilde{u}_0 \in C^{m, \nu}(0, 2T)$, therefore $u_0 \in C^{m, \nu}(0, T]$. Theorem 2.1 is proved.

Further, for $0 \leq t \leq 2T$, the linear homogenous Fredholm equation

$$v(t) = \int_0^{2T} \tilde{K}_0(t, s)v(s)ds, \quad \tilde{K}_0(t, s) = \left. \frac{\partial \tilde{K}(t, s, u)}{\partial u} \right|_{u=\tilde{u}_0(s)}$$

actually takes the form of a linear homogenous Volterra equation,

$$v(t) = \int_0^t \tilde{K}_0(t, s)v(s)ds,$$

and therefore has in $L^\infty(0, 2T)$ only the trivial solution $v \equiv 0$. Thus condition (F3) is fulfilled with respect to the equation (4.8). Finally, let gridpoints (3.4) are used. Then the assertions of Theorem 2.2 follow immediately from the corresponding statements of Theorem 3.2 about equation (4.8).

Thus, Theorem 2.2 is proved.

In a similar way we obtain that the function $\tilde{K}(t, s, u)$ in (4.4) satisfies the conditions (F1') and (F6) whenever the kernel $K(t, s, u)$ of the equation (2.1) satisfies the conditions (V1') and (V6), respectively. Finally we obtain that the function $\tilde{f}(t)$ in (4.6) satisfies (F2') as long as the forcing function $f(t)$ satisfies (V2'), and condition (F3) will be fulfilled with respect to (4.8) as long as equation (2.1) is in $L^\infty(0, T)$ solvable. Therefore, assuming that the conditions of Theorem 2.3, respectively, Theorem 2.4, are fulfilled, the statements of Theorems 2.3 and 2.4 follow immediately from Theorem 3.3..

Hence, Theorems 2.3 and 2.4 are proved.

5. Numerical example

Let $m = 2$ and let $\eta_1 = -\frac{1}{\sqrt{3}}$ and $\eta_2 = \frac{1}{\sqrt{3}}$ be the knots of the Gauss quadrature formula $\int_{-1}^1 \varphi(\xi)d\xi \approx \varphi(\eta_1) + \varphi(\eta_2)$. In this case the approximate solution $u_N \in E_N$ to equation (2.1) in the interval $[t_{j-1}, t_j]$ ($j = 1, \dots, N$) can be represented in the form

$$(5.1) \quad u_N(t) = c_{j1} \frac{\xi_{j2} - t}{\xi_{j2} - \xi_{j1}} + c_{j2} \frac{t - \xi_{j1}}{\xi_{j2} - \xi_{j1}}, \quad t_{j-1} \leq t \leq t_j,$$

where ξ_{j1}, ξ_{j2} are the knots (2.6) and the coefficients $c_{jk} = u_N(\xi_{jk})$ ($k = 1, 2; j = 1, \dots, N$) are determined from the system (2.10) ($m = 2$). From Theorem 2.2 we obtain for the approximate solution (5.1) the estimate

$$(5.2) \quad \varepsilon_N \leq ch^2 \quad \text{for} \quad \begin{cases} r > \frac{1}{1-\nu} & \text{if } \nu \geq 0 \\ r \geq 1 & \text{if } \nu < 0 \end{cases}$$

where $h = T/N$ and

$$(5.3) \quad \varepsilon_N = \max_{i=1,2;j=1,\dots,N} |u_N(\xi_{ji}) - u_0(\xi_{ji})|.$$

From Theorems 2.3 and 2.4, we obtain the following estimates for the error (5.3) (see (2.22), (2.24), $m = 2$ with $\mu = 1$):

$$(5.4) \quad \varepsilon_N \leq ch^2 \begin{cases} h & \text{if } \nu < 0 \\ h(1 + |\log h|) & \text{if } \nu = 0 \\ h^{1-\nu} & \text{if } \nu > 0 \end{cases}$$

or (in conditions of Theorem 2.4)

$$(5.5) \quad \varepsilon \leq ch^2 \begin{cases} h^2 & \text{if } \nu < -1 \\ h^2(1 + |\log h|) & \text{if } \nu = -1 \\ h^{1-\nu} & \text{if } \nu > -1 \end{cases}$$

provided that

$$(5.6) \quad \begin{cases} r \geq 1 & \text{if } \nu < -2 \\ r > \frac{4}{2-\nu} & \text{if } -2 \leq \nu \leq -1 \\ r \geq \frac{3-\nu}{2-\nu} & \text{if } -1 < \nu < 1 - \sqrt{2} \\ r > \frac{2}{1-\nu} & \text{if } 1 - \sqrt{2} \leq \nu < 1 \end{cases}$$

Now we present the following example. Consider the integral equation

$$(5.7) \quad u(t) = \int_0^t (t-s)^{-1/2} u^2(s) ds + t^{1/2} \left(1 - \frac{4}{3}t\right), \quad 0 \leq t \leq T.$$

It is easy check that $u_0(t) = t^{1/2}$ is the exact solution to equation (5.7) and Assumptions (V1') and (V2') of Theorem 2.3 are fulfilled with $\nu = 1/2$, $m = 2$, $\mu = 1$.

The equation (5.7) was solved numerically by the collocation method (2.10) ($m = 2$) where the points $\eta_2 = -\eta_1 = 1/\sqrt{3}$ were used for determining the collocation points (2.6) by $r = 41/10$ (see (5.6)). At every step (on every subinterval $[t_{j-1}, t_j]$) the coefficients $c_{ji} = u_N(\xi_{ji})$ ($i = 1, 2$) were calculated from (2.10) ($m = 2$) by the Newton method. All the integrals which are needed for the construction of the system (2.10) were found analytically. Some of the numerical results by $T = 3/4$ are presented in the following Table 1 where ε_N is defined in (5.3). The experiments were carried out on a computer IBM 4381 (in double precision)

Table 1

N	$\varepsilon_N(r = 41/10)$	$\varepsilon_{N/2}/2^{5/2}$
4	0.55 E - 01	
8	0.80 E - 02	0.97 E - 02
16	0.12 E - 02	0.14 E - 02
32	0.19 E - 03	0.21 E - 03
64	0.32 E - 04	0.33 E - 04
128	0.20 E - 05	0.57 E - 05

From Table 1 we can see that the numerical results are consistent with the theoretical estimate which is $\varepsilon_N = O(h^{5/2})$, $h = 0.75/N$. Notice that the number of collocation points (the number of unknowns) is $2N$.

In the same example on longer intervals (e.g. $T = 4$), same instability of the approximate solution was observed. The numerical stability of the scheme is worth to be examined independently.

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