ALGEBRAIZATIONS OF THE FIRST ORDER LOGIC

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INTRODUCTION

Three approaches of algebraization of the First Order Logic (FOL) are well-known; the first: the Cylindric Algebras by Tarski [15], and the second one: the Polyadic Algebras by Halmos [5] (we will call them in this work as Halmos Algebras (HA)). The third approach has been proposed by Lawvere [7]. It is based on categorical considerations.

There exist many works in which several algebraizations of FOL and their properties were investigated. Ch. Pinter [10] showed that the system of axioms of Halmos Algebras may be simplified in comparison with the system of axioms that was defined by Halmos. One chapter about Cylindric Algebras is devoted in the book of Monk [9]. Connections between ideals of Polyadic Algebras and Monadic Algebras have been studied by F.B. Wright [19]. Note that ideals and filters in Halmos Algebras are defined simply and naturally. A fundamental work of W.Craig is devoted to Boolean Algebras with operators [3]. Later works belong to I.Nemeti [9], H.Andreka [1] and I.Shain [14].

The idea of categorical description of algebraic theories has been formulated by Lawvere [7]. Deducing from this point of view, a certain category called algebraic theory is determined. This category describes a syntax of Algebras of the theory and all the algebras may be considered as a functor from an algebraic theory (category) to the category, (for instance) SETS .

Using these ideas E.Beniaminov [2] has defined special algebraic structures for studying the algebraic models of Relational databases. These algebras were called Relational Algebras (RA). In our opinion, in this case the term "Relational" looks to a certain extent unsuitable, since it "was occupied" by another structure. But in the present work author will use the term "Relational Algebra" in the sense of Beniaminov.

In 1983 Prof. B.Plotkin defined Halmos Algebras and Relational Algebras, which have been specialized in some variety θ (in short, HA_θ and RA_θ, respectively) [11]. This is generalized notion and we can obtain the pure Halmos Algebras and Relational Algebras as a particular case of HA_θ and RA_θ.

While Halmos Algebras and Cylindric Algebras are being thoroughly investigated, Relational Algebras (according to Beniaminov) were not widely known. In particular, important notions of a filter and an ideal of RA-s which were not described earlier, were given by the author [16] and we will discuss this question in chapter 1. Note also that relation between Halmos Algebras and Cylindric Algebras is simple and was described by B.A.Galler [4] in 1957.

The problem of connecting HA-s and RA-s has been stated by B.Plotkin in Riga Algebraic Seminar. It has arisen in connection with the problem of algebraic relational databases model construction. This model was defined by B.Plotkin in 1984 and it can be constructed starting from various algebraizations of the First Order Logic. Therefore it was very important to get the equivalence of different approaches to algebraization of FOL. This equivalence leads to the equivalence of corresponding relational database models. All these problems, as well as several others, are considered in the fundamental work of B.Plotkin [13] .

Note that all the results which we will describe in this work have been derived by the author in 1986 (see [18]). However, all the works in which the main results were described were published in Russian. Therefore the aim of this paper is to present the English version of the Dissertation (which was first published in 1989 in Russian) and some increments to it.

This work consists of three chapters. The first one is devoted to construction of functors from the categories of Halmos Algebras to the categories of Relational Algebras. The questions of relation between axiomatics of HA-s and RA-s are considered in the second chapter. Transition from the categories of RA-s to the categories of HA-s is described in the third chapter. Almost all the results of the paper may be expanded on HA_θ and RA_θ.

I am grateful to my Adviser Prof. B.I. Plotkin for his patient help.

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CHAPTER 1

1.1. The Quantifier Algebras

First we define the scheme of a Halmos Algebra (HA) as a mapping $n{:}I \to \Gamma$ where Γ is a set of sorts of variables. So there is a splitting $I = (I_l, l \in \Gamma)$ and each I_l - is a countable set. Halmos Algebra is a Boolean Algebra (BA) with supplementary operations.

DEFINITION 1.1.

A quantifier (or, more explicitly, an existential quantifier) on BA H is a mapping $\exists{:}H \to H$ such that

E1. $\exists 0 = 0,$

E2. $\exists h > h, h \in H;$

E3. $\exists(h_1 \cap h\ ) = \exists h_1 \cap \exists h\ , h_1, h\ \in H.$

Dually, a universal quantifier on BA H is a mapping $\forall{:}H \to H$ such that

U1. $\forall 1 = 1,$

U2. $\forall h < h, h \in H,$

U3. $\forall(h_1 \cup h_2) = \forall h_1 \cup \forall\, h_2\ h_1, h_2 \in H.$

To each existential quantifier, a universal quantifier can be associated by $\forall h = \overline{\exists \overline{h}}, h \in H.$ On the other hand, $\exists h = \overline{\forall \overline{h}}, h \in H.$ Thus, there exists a one-to-one correspondence between all existential quantifiers and all universal quantifiers on BA-s.

DEFINITION 1.2.

A Quantifier Algebra is a triple $(H, \exists, I),$ where H is BA, $I = (I_l, l \in \Gamma)$ is a system of sets, whose elements are called variables and each I_l is a countable set. Here Γ is a set of sorts and $\exists$ is a mapping from subsets of I to quantifiers on $H,$ satisfying the following conditions:

Q1. $\exists^{\varnothing} h = h, h \in H,$

Q2. $\exists^{J_1} \exists^{J_2} h = \exists^{J_1 \cup J_2} h, h \in H.$

We can define an universal quantifier $\forall^J$ for every quantifier $\exists^J$ setting $\forall^J h = \overline{\exists^J \overline{h}}.$

1.2. The Halmos Algebras

Let, further, for any $l \in \Gamma$, S_l be a semigroup of transformations of I_l into itself and denote by $S = S_1 \times .. \times S_k$ the cartesian product of semigroups $S_1, \ldots, S_k$ where $|\Gamma\ | = k.$

DEFINITION 1.3.

A Halmos Algebra (or a Polyadic Algebra) in scheme $n{:}I \to \Gamma$ is a Quantifier Algebra in scheme $n{:}I \to \Gamma$ for which there is a representation of semigroup S as the semigroup of Boolean endomorphisms of H and for which the following conditions have to be satisfied:

S1. $s_1 \exists^J h = s_2 \exists^J h, h \in H, s_1, s_2 \in S, \quad s_1(\alpha) = s_2(\alpha), \alpha \in \overline{J},\ \overline{J} = I \setminus J.$

S2. $\exists^J sh = s \exists^{s^{-1} J} h$ if s is an injective transformation on $s^{-1} J.$

It is necessary to formulate one more important condition for the representation of semigroup S as the semigroup of Boolean endomorphism of H. We will regard that the identity endomorphism on H corresponds to the identity transformation of S and any monomorphism H corresponds to one-to-one transformation of S. Denote this condition by S3.

DEFINITION 1.4.

Now we can define an identity on HA H as a set of mappings $d=\{d_l, l \in \Gamma\}$ where d_l is a mapping $d_l\colon I_l \times I_l \to H$ and the following axioms hold (for all α, β such that $n(\alpha)=n(\beta)$):

EQ1. $d(\alpha,\alpha)=1$,

EQ2. $sd(\alpha,\beta)=d(s\alpha,s\beta)$,

EQ3. $d(\alpha,\beta)\cap h \le s_\alpha^\beta h$ where s_α^β is replacement such that $s_\alpha^\beta(\gamma)=\gamma$ for all $\gamma \ne \alpha$ and $s_\alpha^\beta(\alpha)=\beta$. We denote $d(\alpha,\beta)=d_l(\alpha,\beta)$ for $\alpha,\beta \in I_l, l \in \Gamma$.

1.3. Supports of elements of a Halmos Algebra

Let H be a Halmos Algebra. To each element $h \in H$ we can associate a special set J which we will call a support. We will say that subset $J \subset I$ is a support of an element $h \in H$ (or h will be supported by J) if $\exists^{\bar{J}} h = h$ where $\bar{J} = I \setminus J$. We will say that an element h is independent of the set J_1 if $\exists^{J_1} h = h$. So a set $J \subset I$ supports an element $h \in H$ if h is independent of $I \setminus J$.

It is important to describe also another "kind" of support. Assume $J \subset I$ and let s,σ be two elements from S such that $s(\alpha)=\sigma(\alpha), \alpha \in J$. The element $h \in H$ is called $s-$supported if $sh=\sigma h$. It is easy to show [11] that if $h \in H$ is $s-$supported by a set J then h is supported by J and vice-versa; if any $h \in H$ is supported by a set J, then h is s-supported by the same J.

The notion of the support is very important and we will frequently use it. For that reason it is useful to get the main properties of supports.

P1. If a set J_1 is a support of some element $h \in H$ then every set J, $J_1 \subset J$ is also a support of h.

P2. If J_1 and J_2 are supports of some element $h \in H$ then the set $J_1 \cap J_2$ is also a support of h.

P3. Assume J is any support of $h \in H$ and let $J_1 \subset I$ be any subset. Then $\exists^J h = \exists^{J \cap J_1} h.$

P4. If an element $\bar{h} \in H$ is supported by any set J then h is also supported by the set J.

P5. Assume a set J_1 is a support of an element h_1 and a set J_2 is a support of an element h_2. Then the set $J_1 \cup J_2$ is a support of both $h_1 \cup h_2$ and $h_1 \cap h_2$.

An element $h \in H$ is called the element with a finite support if there exists a finite set among all the supports of h. Halmos Algebra will be called locally finite iff each element $h \in H$ be supported by some finite set $J \subset I$.

Denote by $\aleph$ the category of locally finite HA-s with an identity.

Denote by $\aleph_1$ the category of Quantifier Algebras over the scheme $n\colon I \to \Gamma$.

Let $\aleph_2$ be a category of Halmos Algebras $(H,\exists,I,G)$ where $G = G_1 \times ... \times G_k$, $|\Gamma| = k$ and each $G_l, l \in \Gamma$ is a group of all one-to-one transformations of the set I_l into itself.

Now we proceed to describe Transformational Algebras over the scheme $n: I \to \Gamma$.

DEFINITION 1.5

A Transformational Algebra is BA for which there exists a representation of S to the semigroup of Boolean endomorphisms of H. So we can say about locally finite algebras (refer to the notion of s-support).

Denote by $\aleph_3$ a category of locally finite Transformational Algebras over the scheme $n: I \to \Gamma$.

Now let us consider several common algebraic notions for Halmos Algebras.

1.4. Ideals and filters in the Halmos Algebras

DEFINITION 1.6.

Assume H_1, H_2 be any HA-s from the category $\aleph$ over the scheme $n: I \to \Gamma$. The mapping $\delta: H_1 \to H_2$ is called HA-s homomorphism if δ is a homomorphism of BA-s H_1, H_2 and δ preserves existentional quantifiers and Boolean endomorphisms on H_1 and H_2, i.e.

H1. $\delta(\exists^J h) = \exists^J \delta h, h \in H_1, J \subset I,$

H2. $\delta(sh) = s\delta h, h \in H_1, s \in S.$

A homomorphism $\delta: H_1 \to H_2$ is called an isomorphism of HA-s if δ is an isomorphism of BA-s H_1, H_2.

Now we will introduce some important notions of ideals and filters of HA-s.

DEFINITION 1.7.

A subset U of any HA H is an ideal of H if:

I1. U is an ideal of the Boolean Algebra H,

I2. If $a \in U$ then for any $J \subset I$, $\exists^J a \in U$,

I3. If $a \in U$ then for any $s \in S$, $sa \in U$.

Similarly, a subset F of the Halmos Algebra H is a filter of H if:

F1. F is a filter of BA H,

F2. If $a \in F$ then, for any $J \subset I$, $\forall^J a \in F$,

F3. If $a \in F$ then $sa \in F$, $s \in S$.

The notions of ideal and filter are dual. So it is clear that $0 \in U$ where U is any ideal of HA. Similarly, if F is a filter of HA H then $1 \in F$. These statements follow from the definitions.

Let $\delta: H_1 \to H_2$ be a homomorphism of HA-s. Consider two sets $U = \{h | h \in H_1 \,\&\, \delta h = 0\}$, $F = \{h | h \in H_1 \,\&\, \delta h = 1\}$. Then U is an ideal of H_1 and F is a filter of H_1. Both U and F are called kernels of δ.

It is a well-known result (see [13]) that any subset $U \subset H$ where H is HA is an ideal of H iff U is an ideal of BA H and $a \in U$ implies $\exists^J a \in U$. Dually, F is a filter of HA H iff F is a filter of BA H and if $b \in F$ implies $\forall^J b \in F$.

1.5. The Halmos Algebra of subsets.

To get HA of subsets over the scheme $n: I \to \Gamma$, let D denote a family of sets, i.e. $D = \{D_l, l \in \Gamma\}$. In other words the family D consists of the sets which may contain elements of different sorts. Denote by $\overline{D}$ the cartesian product of $D_i, i \in I$ i.e. $\overline{D} = \prod D_\alpha, \alpha \in I$, where $D_\alpha = D_{n(\alpha)}$. Then the power set $\tilde{D}$ of the set $\overline{D}$ is BA. For any $A \in \tilde{D}$ and for any $J \subset I$ we can define $\exists^J A$ by the rule: $a \in \exists^J A$ if there exists $b \in A$ such that $a(\alpha) = b(\alpha), \alpha \in I \setminus J$.

Now we can define a representation of the semigroup S to the semigroup of Boolean endomorphisms of H.

First we define the action of any $s \in S$ on $\overline{D}$. For $a \in \overline{D}$ we set $(as)(\alpha) = a(s\alpha)$. Then the action of elements from S on $\tilde{D}$ can be defined by setting

$$sA = \{a | as \in A\}.$$

It was shown (see [13]) that the algebra which has been just defined is really a Halmos Algebra.

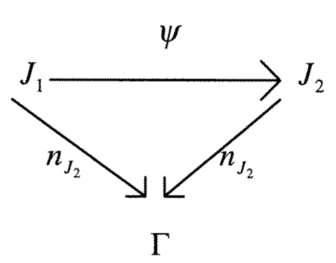
Let us define an identity on $\tilde{D}$ in the following way

$$d(\alpha, \beta) = D_{\alpha,\beta}$$

where $D_{\alpha,\beta}$ is a set of all $a \in \overline{D}$ such that $a(\alpha) = a(\beta)$. It may be shown [13] that all necessary axioms hold. Note that from the definition of a support it follows that, an element $A \subset \overline{D}$ is supported by $J \subset I$ iff from the equation $a(\alpha) = b(\alpha)$ we have that if $a \in A$ then $b \in A$. In other words, if we have to check whether $a \in A$ then we can do it only for $\alpha \in J$, where J is a support of A.

1.6. Relational Algebras

We introduce the notion of the Relational Algebra (RA) by Beniaminov [2]. First we describe the scheme in which RA will be defined. The scheme of RA is a certain category K . Objects of K are mappings of the form $n_J: J \to \Gamma$ where J is a finite set, Γ is a set of sorts. Morphism of two objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ is a mapping $\psi: J_1 \to J_2$ such that the following diagram



is commutative. (We suppose that K does not contain empty objects). To each object $n_J : J \to \Gamma$ from K one can assign Boolean Algebra *R(J),* while to each morphism $\psi : J_1 \to J_2$ correspond two mappings

$$\psi_* : R(J_1) \to R(J_2) \text{ and } \psi^* : R(J_2) \to R(J_1)$$

where ψ_* is a homomorphism of BA-s and ψ^* is such a mapping that for every $a, b \in R(J_2)$ there holds $\psi^*(a \cup b) = \psi^* a \cup \psi^* b.$

We are going to define one more operation on the Relational Algebras denoted by $\times$. Given objects $n_{J_1} : J_1 \to \Gamma$ and $n_{J_2} : J_2 \to \Gamma$ denote by $J_1 \coprod J_2$ the co-union of the sets J_1 and J_2 . Furthermore, let

$$\varepsilon_{J_1} : J_1 \to J_1 \coprod J_2 \text{ and } \varepsilon_{J_2} : J_2 \to J_1 \coprod J_2$$

be the canonical morphisms of the co-union. So, to any two elements $a \in R(J_1)$ and $b \in R(J_2)$, there is a new element $a \times b$ which can be defined by

$$a \times b = (\varepsilon_{J_1})_* a \cap (\varepsilon_{J_2})_* b.$$

DEFINITION 1.8.

Let $n_J : J \to \Gamma$ be an object of K . Then a system $\Re$ of BA-s is called a Relational Algebra over K if the following axioms are satisfied:

R1. $\psi_* : R(J_1) \to R(J_2)$ is a homomorphism of BA-s,

R2. $\psi^* : R(J_2) \to R(J_1)$ is a mapping which preserves operation of disjunction.

R3. a. $(\psi_1 \psi_2)_* = \psi_{1*} \psi_{2*}$, $(\psi_1 \psi_2)^* = \psi_2{}^* \psi_1{}^*$ for any ψ_1, ψ_2,

b. $(1_J)_*, (1_J)^* : R(J) \to R(J)$ are identity mappings,

R4. $\psi_* \psi^* a > a$, $\psi^* \psi_* b < b$ for all elements $a \in R(J_1)$ and $b \in R(J_2)$,

R5. Given morphisms $\psi : J_1 \to J_2$ and $1_{J_3} : J_3 \to J_3$, denote by $(\psi \coprod 1_{J_3}) : J_1 \coprod J_3 \to J_2 \coprod J_3$ the co-union of the morphisms. Then for any $a \in R(J_1)$ and $b \in R(J_2)$ the following holds:

$$(\psi \coprod 1_{J_3})^* (a \times b) = \psi^* a \times b .$$

We make some remarks about the definition of Relational Algebras. A Relational Algebra is a functor from category K to the category of Boolean Algebras *BOOL.* In fact, we deal with two functors. One of them is covariant and the second one is contravariant. As it was shown by the author (with V.Sustavova) [17], axiom R5 may be generalized. More precisely, by using the axioms R1 - R5 we will prove in chapter II that the following axiom holds:

R6. Given morphisms $\psi: J_1 \to J_2$ and $\varphi: J_3 \to J_4$, denote by $(\psi \coprod \varphi): J_1 \coprod J_3 \to J_2 \coprod J_4$ a co-union of these morphisms. Then for any $a \in R(J_2)$ and $b \in R(J_4)$, we have

$$(\psi \coprod \varphi)^*(a \times b) = \psi^* a \times \varphi^* b$$

It is also easy to check (see chapter II) that under the above-mentioned conditions, for any $c \in R(J_1)$ and $d \in R(J_2)$, the following always holds:

$$(\psi \coprod \varphi)_*(c \times d) = \psi_* c \times \varphi_* d.$$

Now we proceed to describe a definition of pure RA-s introduced by B.Plotkin (see [12] and [13]).

Let us consider the following addition properties of category-scheme K. Assume that $n_J: J \to \Gamma$ is any object where $J = \{\alpha_1, \dots, \alpha_n\}$. Then there exist special projections $\delta_J^k: J \to \alpha_k$, for all k, forcing J to be a product $J = \alpha_1 \times \dots \times \alpha_n$. So the Relational Algebra now is a contravariant functor $\mathfrak{R}$: K $\to BOOL$ such that the following axioms satisfied:

M1. For any morphism $\psi: J_1 \to J_2$ the Boolean homomorphism $\psi_* = R(\psi)$ is considered as a functor between order categories $R(J_2)$ and $R(J_1)$, and ψ_* admits the left adjoint $\psi^*: R(J_1) \to R(J_2)$,

M2. For all $\psi: J_1 \to J_2$ and $\varphi: J_3 \to J_4$, $a \in R(J_1)$ and $b \in R(J_3)$

$$(\psi \times \varphi)^*(a \times b) = \psi^* a \times \varphi^* b$$

where $\psi \times \varphi$ is the natural product map, and

$$a \times b = (\pi_{J_1})_* a \cap (\pi_{J_3})_* b,$$

and $\pi_{J_1}: J_1 \times J_3 \to J_1$ $\pi_{J_3}: J_1 \times J_3 \to J_3$ are projections.

It is easy to show that the axiom M1 and axioms R1-R4 are equivalent (if we consider K according to B.Plotkin [13]). On the other hand the axiom M2 regulates a relation between operations $\times$ and *.

At last, it is necessary to describe two additional claims of RA.

The first one is the following. Assume that $I = \{I_l, l \in \Gamma\}$ is a system of sets and let every I_l be a countable set. Suppose, further, $\wp_F(I)$ to be a set of all the finite subsets of I. We will claim that for any object $n_J: J \to \Gamma$, there exists $J \in \wp_F(I)$ and for any $J \in \wp_F(I)$ there exists an object $n_J: J \to \Gamma$.

Before describing the second claim let us make some remarks. Let $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ be objects of the category-scheme K. For any object $n_J: J \to \Gamma$ denote by *sort(J)* the set of all the elements $\gamma \in \Gamma$, such that $n_J(\alpha) = \gamma, \alpha \in J$, i.e.

$$sort(J) = \{\gamma \in \Gamma | n_J(\alpha) = \gamma, \alpha \in \Gamma\}.$$

A morphism $\psi: J_1 \to J_2$ is said to be compatible if the equality $sort(J_1)$ =$sort(J_2)$ holds. Assume now that for the morphism $\psi: J_1 \to J_2$, the insertion $sort(J_1) \subset sort(J_2)$ takes place. It is easy to see that any morphism $\psi: J_1 \to J_2$ can be always written in the $\psi = \varepsilon_{J_1}^{J_2} \varphi$ where φ is the corresponding compatible morphism and $\varepsilon_{J_1}^{J_2}: J_1 \to J_2$ is an identity morphism.

PROPOSITION 1.1.

1. Let $\psi: J_1 \to J_2$ be any injective compatible morphism. Then $\psi_*: R(J_1) \to R(J_2)$ is a monomorphism of BA-s and $\psi^*: R(J_2) \to R(J_1)$ is an epimorphic mapping of BA-s.

2. Assume that $\psi: J_1 \to J_2$ is an epimorphic compatible morphism. Then $\psi_*: R(J_1) \to R(J_2)$ is an epimorphism of BA-s and $\psi^*: R(J_2) \to R(J_1)$ is an injective mapping of BA-s.

PROOF 1. First for $\psi: J_1 \to J_2$ we make an opposite morphism $\varphi: J_2 \to J_1$ by setting $\varphi\beta = \alpha$ for every $\alpha \in J_1$ such that $\psi(\alpha) = \beta$. For the remaining elements $\beta \in J_2$ let φ act arbitrarily. We have $\varphi\psi = 1_{J_1}$ and then using the axiom R3 we get $\varphi_*\psi_* = (1_{J_1})_*$ and $\psi^*\varphi^* = (1_{J_1})^*$. Since $(1_{J_1})_*$ and $(1_{J_1})^*$ are both identity mappings of BA $R(J_1)$, the item 1 is proved.

2. The proof resembles the one just discussed. We just have to construct an injective φ for ψ . The rest of the proof is the same •

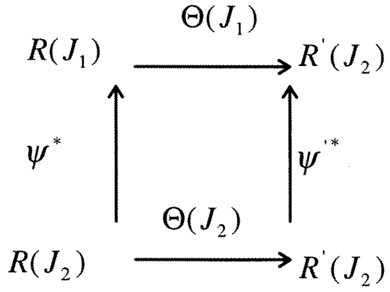
And now we are ready to formulate the second claim. Assume that $\psi: J_1 \to J_2$ is any injective morphism such that $sort(J_1) \subset sort(J_2)$. We will claim that $\psi_*: R(J_1) \to R(J_2)$ and $\psi^*: R(J_2) \to R(J_1)$ which correspond to $\psi: J_1 \to J_2$ are monomorphism of BA-s and epimorphic mapping, respectively. We will denote the condition (axiom) by R6.

1.7. Homomorphisms of Relational Algebras

DEFINITION 1.9.

Let R_1, R' be RA-s over the scheme-category K and let $\Theta = \Theta(J)$ be a set of homomorphisms of BA-s $\Theta(J): R(J) \to R'(J)$ for each object $n_J: J \to \Gamma$ of K. We will say Θ is a homomorphism of RA R_1, R' if each $\Theta(J)$ acts in accordance to the corresponding mappings for morphism $\psi: J_1 \to J_2$, i.e. the following diagrams

$$\begin{array}{ccc} R(J_1) & \xrightarrow{\Theta(J_1)} & R'(J_2) \\ \psi_* \downarrow & & \downarrow \psi'_* \\ R(J_2) & \xrightarrow{\Theta(J_2)} & R'(J_2) \end{array}$$



are commutative. In other words, for any $a \in R(J_1)$, $b \in R(J_2)$, we have

$$\Theta(J_2)\psi_* a = \psi'_* \Theta(J_1)a, \quad \psi'^* \Theta(J_2)b = \Theta(J_1)\psi^* b.$$

A homomorphism Θ between RA-s is an isomorphism if for every object $n_J: J \to \Gamma$ of K the homomorphism $\Theta(J)$ is actually an isomorphism of BA-s. In fact, it is easy to see a homomorphism of Relational Algebras is a natural transformation of the functors.

1.8. Ideals and filters in Relational Algebras

DEFINITION 1.10.

Let R be RA over the scheme K . Assume that $I=\{I_J\}$ is a system of ideals of BA $R(J)$ for every object $n_J: J \to \Gamma$. The system I is an ideal of a Relational Algebra R if for any morphism $\psi: J_1 \to J_2$ the following conditions are satisfied:

I1. $\psi_* I_{J_1} \subset I_{J_2}$ where $\psi_* I_{J_1} = \{\psi_* a | a \in I_{J_1}\}$,

I2. If $a-b \in I_{J_2}$ then $\psi^* a - \psi^* b \in I_{J_1}$, where $a-b = \bar{a} \cap b$.

Dually, let $F=\{F_J\}$ be a system of filters of BA $R(J)$ for every object $n_J: J \to \Gamma$. A system F is a filter of RA R if for any morphism $\psi: J_1 \to J_2$ the following conditions hold:

F1. $\psi_* F_{J_1} \subset F_{J_2}$,

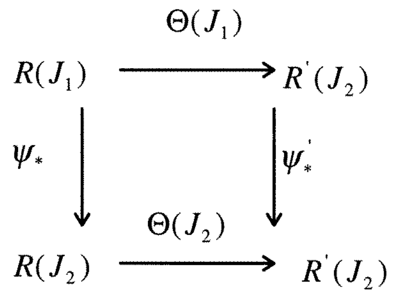
F2. If $a \to b \in F_{J_2}$ then $\psi^* a \to \psi^* b \in F_{J_1}$ where $a \to b = \bar{a} \cup b$.

We have to check that the above notions are well defined. We will carry out it only for filters because the necessary proof for ideals is the same as for filters. Let

$\Theta: R \to R'$ be a homomorphism of RA-s. Denote $F_{\ker} = \{F_J\}$ a system of kernels of homomorphisms $\Theta(J)$ for every object $n_J: J \to \Gamma$, i.e. $\{F_J\} = \ker \Theta(J)$.

THEOREM 1.1. Assume $\Theta: R \to R'$ is a homomorphism of RA-s. Then the system $F_{\ker} = \{F_J\}$ of kernels is a filter of RA R.

PROOF. Assume $\psi: J_1 \to J_2$ is any morphism. First it is necessary to show that $\psi_* F_{J_1} \subset F_{J_2}$. Let $a \in F_{J_1}$ be arbitrary element. Using the following commutative diagram (see the definition of a homomorphism of RA-s)



we get $\Theta(J_2)\psi_* a = \psi'_* \Theta(J_1)a$. Because $a \in F_{J_1}$, $\{F_J\} = \ker \Theta(J)$, it follows that $\Theta(J_1)a = 1_{\Re'(J_1)}$ where $1_{\Re'(J_1)}$ is the unit of BA $R'(J_1)$. Hence $\psi'_* \Theta(J_1)a = 1_{\Re'(J_2)}$ and therefore $\Theta(J_2)\psi_* a = 1_{\Re'(J_2)}$. This implies $\psi_* a \in F_{J_2}$.

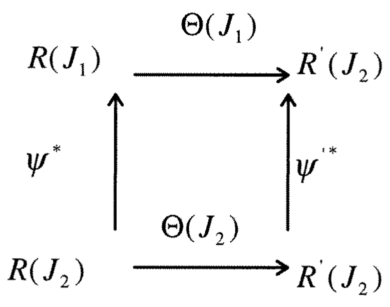
Now it is necessary to show that $a \to b \in F_{J_2}$ implies $\psi^* a \to \psi^* b \in F_{J_1}$. Remember that for any $a, b \in R(J)$ the following takes place:

$$(a \to b) = 1 \Leftrightarrow a < b .$$

So assume that $a \to b \in F_{J_2}$. Then we obtain

$$\Theta(J_2)(a \to b) = 1_{\Re'(J_2)} \Rightarrow (\Theta(J_2)a \to \Theta(J_2)b) = 1_{\Re'(J_2)} \Rightarrow \Theta(J_2)a < \Theta(J_2)b.$$

Since $\Theta(J_2)a < \Theta(J_2)b$, it follows $\psi'^* \Theta(J_2)a < \psi'^* \Theta(J_2)b$. Using the commutative diagram



we get $\psi'^{*}\Theta(J_2)a < \psi'^{*}\Theta(J_2)b \Rightarrow \Theta(J_1)\psi^{*}a < \Theta(J_1)\psi^{*}b.$ This implies $\Theta(J_1)\psi^{*}a \to \Theta(J_1)\psi^{*}b = 1_{\Re'(J_1)}$ and therefore

$$\Theta(J_1)(\psi^{*}a \to \psi^{*}b) = 1_{\Re'(J_1)}, \text{ i.e. } \psi^{*}a \to \psi^{*}b \in F_{J_1} \bullet$$

We introduce the following notations.

First denote by $\Re$ the category of RA over K. Denote, further, by K_1 a subcategory of K in which we will consider the morphisms $\varepsilon_{J_1}^{J_2}: J_1 \to J_2$ where $J_1 \subset J_2$. Denote by $\Re_1$ the category of RA-s over the scheme K_1.

Consider K_2 as a subcategory of K having only injective morphisms. Denote by $\Re_2$ a category of RA-s over the scheme K_2.

And, finally, let $\Re_3$ be category of RA-s over the scheme K, and suppose that for every algebra from $\Re_3$ only a homomorphism ψ_* of BA-s for any morphism $\psi: J_1 \to J_2$ is defined.

1.9 Transition from the category of the Halmos Algebras To the category of the Relational Algebras

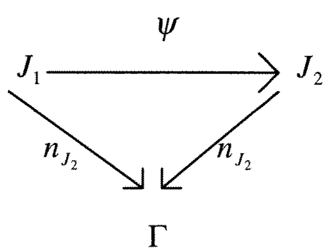
Now we would like to describe the construction of the functor *rel* from the category of HA $\aleph$ to the category of RA $\Re$. It clear that the functors

$$rel_1: \aleph_1 \to \Re_1,$$
$$rel_2: \aleph_2 \to \Re_2,$$
$$rel_3: \aleph_3 \to \Re_3$$

will arise naturally from *rel.* Unfortunately, we do not have here the possibility to give a detailed description of the construction of the corresponding functors. Therefore we will consider only general scheme of the constructions of the functor *rel.* The Reader can find the omitted proofs in [13] and [18].

Let $H \in Ob\,\aleph$. We are going to associate with H the Relational Algebra *relH.* First, on the bases of the scheme $n: I \to \Gamma$ we will construct the scheme - category K. To do this, we consider all the finite subsets $J \subset I$ and all the restrictions of the mapping $n: I \to \Gamma$ on these finite $J \subset I$.

These restrictions will be the objects of the category- scheme K. The morphism ψ of two objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ is a mapping $\psi: J_1 \to J_2$ such that the diagram



is commutative.

Now we associate to every object $n_J: J \to \Gamma$ the Boolean Algebra *H(J)*.

Let *H(J)* be a set of all elements of *H* such that $\exists^{\bar{J}} h = h$, i.e. every $h \in H(J)$ is supported by *J*. It is a well-known fact [13] that *H(J)* is a BA.

So for every object $n_J: J \to \Gamma$ from K we have a BA *H(J)*. For any two objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ let us denote by ψ morphism $\psi: J_1 \to J_2$ and let $s \in S$ be an element such that $s\alpha = \psi\alpha$, $\alpha \in J_1$. Here *S* is the corresponding semigroup.

DEFINITION 1.11. For every $h \in H(J_1)$ we set

$$\psi_* h = sh.$$

All the necessary properties of the definition considered in [18]. Now we have to construct ψ^* for any morphism $\psi: J_1 \to J_2$. Let $J_1 = \{\alpha_1, \ldots, \alpha_n\}$. Consider case $J_1 \cap J_2 = \varnothing$. We set for any $h \in H(J_2)$,

$$\psi^* h = \exists^{J_2} (h \cap d(\alpha_1, \psi\alpha_1) \cap \ldots \cap d(\alpha_n, \psi\alpha_n)).$$

General case and all the necessary proofs are given in [18].

From the scheme $n: I \to \Gamma$ we have constructed the category-scheme K. Then for every object $n_J: J \to \Gamma$ we obtained BA *H(J)* and for any morphism $\psi: J_1 \to J_2$ we have two mappings $\psi_*: H(J_1) \to H(J_2)$ and $\psi^*: H(J_2) \to H(J_1)$. Furthermore, the mapping ψ_* is a homomorphism of the corresponding BA-s and the mapping ψ^* preserves the operation of the disjunction of BA-s. Denote by *relH* all these BA-s over the scheme-category and all mappings ψ_* and ψ^* that correspond to morphisms from K.

THEOREM 1.2. Let *H* be HA over the scheme $n: I \to \Gamma$ and let *relH* be a construction which was mentioned above . Then *relH* is a Relational Algebra over the scheme K.

By using this theorem we can construct a functor from the category of Halmos Algebras $\aleph$ to the category of Relational Algebras $\Re$.

Let H_1, H_2 be any algebras from the category $\aleph$ over the scheme $n: I \to \Gamma$ and let $\mu: H_1 \to H_2$ be a homomorphism of HA-s. Denote by R_1, R_2 the Relational Algebras $rel(H_1)$ and $rel(H_2)$ correspondingly. Note that the homomorphism μ preserves supports of the elements from H_1 and therefore μ induces homomorphisms $\mu(J): R_1(J) \to R_2(J)$ for every object $n_J: J \to \Gamma$ from the category K. Denote $rel(\mu) = \theta$ where $\theta = \{\theta(J)\}$ for every object $n_J: J \to \Gamma$ from K. It is easy to show (see [18]) that $rel(\mu)$ is a homomorphism of the corresponding RA.

CHAPTER 2

OPERATIONS OF HALMOS ALGEBRAS IN RELATIONAL ALGEBRAS

In this chapter we will show that Boolean Algebras *R(J)* (considered together with the semigroup S_J) of RA-s from the various categories may be presented as new algebraic structures over the scheme $n_J: J \to \Gamma$.

First, it is necessary to consider some elementary properties of a Relational Algebra. These properties are useful for the further proofs. They were investigated by W.Craig (see [3]). We will use them frequently without citation.

PROPOSITION 2.1 Let $\psi: J_1 \to J_2$ be a morphism of two arbitrary objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ from the category-scheme K, and assume that $a \in R(J_1), b \in R(J_2)$. Then:

1. $\psi^* \psi_* \psi^* a = \psi^* a$,
2. $\psi_* \psi^* \psi_* b = \psi_* b$,
3. If $a < \psi_* b$ then $\psi^* a < b$,
4. If $\psi^* a < b$ then $a < \psi_* b$,
5. $\psi^* a \cap b = 0$ if and only if $a \cap \psi_* b = 0$.

PROOF. 1. According to the axioms of RA-s for $a \in R(J_2)$ we have $\psi_* \psi^* a > a$ which implies $\psi^* \psi_* \psi^* a > \psi^* a$. On the other hand, if we denote $\psi^* a = c$, then we can write (using the axioms of a RA)

$$\psi^* \psi_* c < c \Rightarrow \psi^* \psi_* \psi^* a < \psi^* a \text{ and finally } \psi^* \psi_* \psi^* a = \psi^* a.$$

2. For $b \in R(J_2)$ we get $\psi^* \psi_* b < b \Rightarrow \psi_* \psi^* \psi_* b < \psi_* b$. Then denote $\psi_* b = d$. We have $\psi_* \psi^* d > d \Rightarrow \psi_* \psi^* \psi_* b > \psi_* b$ and thus, $\psi_* \psi^* \psi_* b = \psi_* b$.

3. Assume that $a < \psi_* b$. Then $\psi^* a < \psi^* \psi_* b$ and $\psi^* a < b$.

4. Assume that $\psi^* a < b$. Then $\psi_* \psi^* a < \psi_* b$ and $a < \psi_* b$.

5. Assume that $\psi^* a \cap b = 0$. Then we have

$\psi^* a \cap b = 0 \Leftrightarrow \psi^* a < \bar{b} \Leftrightarrow a < \psi_* \bar{b} \Leftrightarrow a < \overline{\psi_* b} \Leftrightarrow a \cap \psi_* b$ •

PROPOSITION 2.2. 1. Let $\psi: J_1 \to J_2$ be an injective morphism. Then $\psi^* \psi_* a = a$, $a \in R(J_1)$.

2. If $\psi: J_1 \to J_2$ is a bijective morphism then $\psi_* \psi^* b = b$, $b \in R(J_2)$.

PROOF. 1. By proposition 1.1 (item 2) we can write $\psi_* \psi^* \psi_* a = \psi_* a$. Since ψ is an injective morphism, ψ_* is a monomorphism of the corresponding BA-s (see axiom R1). Therefore, $\psi^* \psi_* a = a$.

2. Proof is the same as in item 1, it is only necessary to use item 1 of proposition 1.1. •

Now we are going to consider the category $\mathfrak{R}_1$ of the Relational Algebras over the scheme-category K_1 in which morphisms for any objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ are defined only when $J_1 \subseteq J_2$ and they are identity injections and look as $\varepsilon_{J_1}^{J_2}: J_1 \to J_2$. Let us fix a Relational Algebra R from the category $\mathfrak{R}_1$.

THEOREM 2.1. Let R be a Relational Algebra over the scheme-category K_1. Then any BA *R(J)* of the Relational Algebra R may be transformed to a Quantifier Algebra over the scheme $n: J \to \Gamma$.

We will divide the proof of the theorem in several lemmas and propositions. First, we define existential quantifier $\exists_J^{J_1} : R(J) \to R(J)$ of the Boolean Algebra *R(J)* for any proper subset $J_1 \subset J$.

DEFINITION 2.1. For any $a \in R(J)$ we set

$$\exists_J^{J_1} a = (\varepsilon_{J_2}^J)_* (\varepsilon_{J_2}^J)^* a ,$$

where $J_2 = J / J_1$.

Now we need to show that this is well defined, i.e. we have to check that for any $a, b \in R(J)$, the following axioms hold:

1. $\exists_J^{J_1} \varnothing = \varnothing$,
2. $\exists_J^{J_1} a > a$,
3. $\exists_J^{J_1} (a \cap \exists_J^{J_1} b) = \exists_J^{J_1} a \cap \exists_J^{J_1} b$.

We use the following lemma, by W.Craig [2].

LEMMA 2.1. Let $\psi : J_1 \to J_2$ be a morphism. Then for any $a \in R(J_2)$, $b \in R(J_1)$ there holds:

$$\psi^* (a \cap \psi_* b) = \psi^* a \cap b .$$

PROOF. We have $\psi^* (a \cap \psi_* b) < \psi^* a$ and $\psi^* (a \cap \psi_* b) < \psi^* \psi_* b < b$. It implies

$$\psi^* (a \cap \psi_* b) < \psi^* a \cap b$$

On the other hand,

$$a \cap \psi_* b < \psi_* \psi^* (a \cap \psi_* b) \Rightarrow a \cap \psi_* b \cap \psi_* \psi^* (a \cap \psi_* b) = \varnothing \Rightarrow$$
$$\Rightarrow a \cap \psi_* (b \cap \overline{\psi^* (a \cap \psi_* b)}) = \varnothing \Rightarrow (\psi^* a \cap b) \cap \overline{\psi^* (a \cap \psi_* b)} = \varnothing \Rightarrow$$
$$(\psi^* a \cap b) < \psi^* (a \cap \psi_* b) \bullet$$

PROPOSITION 2.3. For any proper subset $J_1 \subset J_2$ the mapping

$$\exists_J^{J_1} : R(J) \to R(J)$$

is an existential quantifier of BA *R(J)*.

PROOF. Let us consider three conditions for an existential quantifier which were pointed above. It is easy to see that the first and the second conditions hold. This follows from axioms R3 and R4 of the definition of a Relational Algebra and from the fact that for any morphism ψ both mappings ψ_* and ψ^* preserve zero elements of BA-s. So, let us check the third condition. We have

$$\exists_J^{J_1} (a \cap \exists_J^{J_1} b) = (\varepsilon_{J_2}^J)_* (\varepsilon_{J_2}^J)^* [a \cap (\varepsilon_{J_2}^J)_* (\varepsilon_{J_2}^J)^* b] =$$
$$= (\varepsilon_{J_2}^J)_* [(\varepsilon_{J_2}^J)^* a \cap (\varepsilon_{J_2}^J)^* b]$$

Here we used lemma 2.1 and item 1 of proposition 2.1. Since $(\varepsilon_{J_2}^J)_*$ is a homomorphism of BA-s, we get

$$(\varepsilon_{J_2}^J)_* [(\varepsilon_{J_2}^J)^* a \cap (\varepsilon_{J_2}^J)^* b] = (\varepsilon_{J_2}^J)_* (\varepsilon_{J_2}^J)^* a \cap (\varepsilon_{J_2}^J)_* (\varepsilon_{J_2}^J)^* b] =$$

$$= \exists_J^{J_1} a \cap \exists_J^{J_1} b \bullet$$

Hence, the mapping $\exists_J^{J_1}$ defined this way is really an existential quantifier for any proper subset $J_1 \subset J$.

Now we would like to show that for any BA $R(J)$ from an arbitrary RA, the following axioms of a Quantifier Algebra hold:

Q1. $\exists_J^{\varnothing} a = a, a \in R(J)$;

Q2. $\exists_J^{J_1} \exists_J^{J_2} a = \exists_J^{J_1 \cup J_2} a$, where J_1 , J_2 , $J_1 \cup J_2$ - are proper subsets of the set J, $a \in R(J)$.

It is easy to understand that the first condition takes place. Indeed, by definition 1.1 we get

$$\exists_J^{\varnothing} a = (\varepsilon_J^J)_* (\varepsilon_J^J)^* a = (1_J)_* (1_J)^* a = a .$$

We will divide the proof of the second axiom in three propositions.

PROPOSITION 2.4. Assume $J_1 \subset J_2 \subset J$. Then for every $a \in R(J)$

$$\exists_J^{J_1} \exists_J^{J_2} a = \exists_J^{J_1 \cup J_2} a = \exists_J^{J_2} a .$$

PROOF. Denote $J_3 = J \setminus J_1$, $J_4 = J \setminus J_2$. It is clear that $J_4 \subset J_3$. Then we have

$$\exists_J^{J_1} \exists_J^{J_2} a = (\varepsilon_{J_3}^J)_* (\varepsilon_{J_3}^J)^* (\varepsilon_{J_4}^J)_* (\varepsilon_{J_4}^J)^* a =$$

$$= (\varepsilon_{J_3}^J)_* (\varepsilon_{J_3}^J)^* [(\varepsilon_{J_3}^J \cdot \varepsilon_{J_4}^{J_3})_*] (\varepsilon_{J_4}^J)^* a =$$

$$= (\varepsilon_{J_3}^J)_* (\varepsilon_{J_3}^J)^* [(\varepsilon_{J_3}^J)_* \cdot (\varepsilon_{J_4}^{J_3})_*] (\varepsilon_{J_4}^J)^* a =$$

$$(\varepsilon_{J_3}^J)_* [(\varepsilon_{J_3}^J)^* (\varepsilon_{J_3}^J)_*] (\varepsilon_{J_4}^{J_3})_* (\varepsilon_{J_4}^J)^* a = (\varepsilon_{J_3}^J)_* (\varepsilon_{J_4}^{J_3})_* (\varepsilon_{J_4}^J)^* a =$$

$$= (\varepsilon_{J_3}^J \cdot \varepsilon_{J_4}^{J_3})_* (\varepsilon_{J_4}^J)^* a = (\varepsilon_{J_4}^J)_* (\varepsilon_{J_4}^J)^* a = \exists_J^{J_2} a \bullet$$

Now it is useful to write the fifth axiom (more exactly, the special case of it) of a Relational Algebra before the next proposition.

Assume $n_A : A \to \Gamma$, $n_B : B \to \Gamma$, $n_C : C \to \Gamma$, $n_D : D \to \Gamma$ and let $n_E : E \to \Gamma$ be any objects of the category K such that $A \cap C = \varnothing$, $B \cap C = \varnothing$, $A \cup C = D$, $B \cup C = E$. Suppose that $s_1 : A \to B$ is a morphism, and let $s : D \to E$ be such morphism that $s\alpha = s_1\alpha$ if $\alpha \in A$ and $s\alpha = \alpha$ for $\alpha \in C$. In these conditions for every $a \in R(B)$, $b \in R(C)$, the following equation holds:

$$s^* [(\varepsilon_B^E)_* a \cap (\varepsilon_c^E)_* b] = (\varepsilon_A^D)_* s_1^* a \cap (\varepsilon_C^D)_* b ,$$

or equivalently

$$s^* (a \times b) = s_1^* a \times b .$$

PROPOSITION 2.5. Assume that $J_1 \subset J$, $J_2 \subset J$ and $J_1 \cap J_2 = \varnothing$. Then

$$\exists_J^{J_1} \exists_J^{J_2} a = \exists_J^{J_1 \cup J_2} a , \ a \in R(J).$$

PROOF. Denote $J_3 = J \setminus J_1$, $J_4 = J \setminus J_2$. By the definition we get

$$\exists_J^{J_1} \exists_J^{J_2} a = (\varepsilon_{J_3}^J)_* (\varepsilon_{J_3}^J)^* (\varepsilon_{J_4}^J)_* (\varepsilon_{J_4}^J)^* a ,$$

$$\text{and} \quad \exists_J^{J_1 \cup J_2} a = (\varepsilon_{J_3 \cap J_4}^J)_* (\varepsilon_{J_3 \cap J_4}^J)^* a .$$

Note here that for arbitrary $J' \subset J$ we can always write

$$(\varepsilon_{J'}^J)_* (\varepsilon_{J'}^J)^* a = (\varepsilon_{J'}^J)_* (\varepsilon_{J'}^J)^* a \cap (\varepsilon_{J''}^J)_* (\varepsilon_{J''}^J)^* 1 ,$$

where $J''=J \setminus J'$, 1 is the unit of the Boolean Algebra $R(J)$ and $\times$ - is an operation which was defined earlier (see the definition of *RA*). So we obtain

$$\exists_J^{J_1} \exists_J^{J_2} a = (\varepsilon_{J_3}^J)^* [(\varepsilon_{J_4}^J)^* a \times (\varepsilon_{J_2}^J)^* 1] \times (\varepsilon_{J_1}^J)^* 1 .$$

Let us consider the expression

$$(\varepsilon_{J_3}^J)^* [(\varepsilon_{J_4}^J)^* a \times (\varepsilon_{J_2}^J)^* 1] .$$

We are ready to use the fifth axiom here. Really, if we consider $(\varepsilon^J_{J_2})^*$, J_3, $J_3\cap J_4$, J_4 and J_2 instead of s^*, D, A, B and C, respectively, then we get

$$(\varepsilon^J_{J_3})^*[(\varepsilon^J_{J_4})^* a\times(\varepsilon^J_{J_2})^* 1]=(\varepsilon^{J_4}_{J_3\cap J_4})(\varepsilon^J_{J_4})^* a\times(\varepsilon^J_{J_2})^* 1=$$
$$=(\varepsilon^J_{J_4}\cdot\varepsilon^{J_4}_{J_3\cap J_4})^* a\times(\varepsilon^J_{J_2})^* 1=(\varepsilon^J_{J_3\cap J_4})^* a\times(\varepsilon^J_{J_2})^* 1=$$
$$=(\varepsilon^{J_3}_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a\cap(\varepsilon^{J_3}_{J_2})_*(\varepsilon^J_{J_2})^* 1=(\varepsilon^{J_3}_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a\ .$$

Thus

$$\exists^{J_1}_J\exists^{J_2}_J a=(\varepsilon^{J_3}_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a\times(\varepsilon^J_{J_1})^* 1=(\varepsilon^J_{J_3})_*(\varepsilon^{J_3}_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a\cap$$
$$\cap(\varepsilon^J_{J_1})_*(\varepsilon^J_{J_1})^* 1=(\varepsilon^J_{J_3}\cdot\varepsilon^{J_3}_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a\cap(\varepsilon^J_{J_1})_*(\varepsilon^J_{J_1})^* 1=$$
$$=(\varepsilon^J_{J_3\cap J_4})_*(\varepsilon^J_{J_3\cap J_4})^* a=\exists^{J_1\cup J_2}_J a\ \bullet$$

Now it is necessary to complete the definition of quantifier $\exists^{J_1}_J$ for $J_1=J$.

DEFINITION 2.2. Let $n_J: J\to\Gamma$ be an object of the category K and assume $n_{J_1}: J_1\to\Gamma$ be an object such that $J\subset J_1$. We set

$$\exists^J_J a=(\varepsilon^{J_1}_J)^*(\varepsilon^{J_1}_{J_2})_*(\varepsilon^{J_1}_{J_2})^*(\varepsilon^{J_1}_J)_* a\ ,$$

where $J_2=J_1\setminus J$.

Let us make some remarks about this definition. If we will define the quantifier $\exists^J_J$ according to definition 2.1 then we get $\exists^J_J a=(\varepsilon^J_\varnothing)_*(\varepsilon^J_\varnothing)^* a$, $a\in R(J)$, but there does not exist the empty object $n_\varnothing:\varnothing\to\Gamma$ in the scheme-category K_1 (see the definition of a Relational Algebra), therefore there does not exist the Boolean Algebra $R(\varnothing)$ in RA R. That is why it is necessary, first, to embed the Boolean Algebra *R(J)* into the Boolean Algebra $R(J_1)$ by the monomorphism $(\varepsilon^{J_1}_J)_*$ and then by the usual way we can define the quantifier $\exists^J_{J_1}$. After that we return to BA *R(J)* using the mapping $(\varepsilon^{J_1}_J)^*$. But this definition requires the proof of its independence from the choice of J_1.

PROPOSITION 2.6. The definition 2.2 does not depend on the choice of J_1.

PROOF. Assume $n_{J'}: J'\to\Gamma$, $n_{J'}: J'\to\Gamma$ be any objects such that $J\subset J'$, $J\subset J''$. We have to prove that

$$(\varepsilon^{J'}_J)^*(\varepsilon^{J'}_{J_1})_*(\varepsilon^{J'}_{J_1})^*(\varepsilon^{J'}_J)_* a=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_J)_* a\ ,$$

where $J_1=J'\setminus J$, $J_2=J''\setminus J$, $a\in R(J)$. Examine two cases.

CASE I. Assume $J'\subset J''$. We get

$$(\varepsilon^{J'}_J)^*(\varepsilon^{J'}_{J_1})_*(\varepsilon^{J'}_{J_1})^*(\varepsilon^{J'}_J)_* a=$$
$$=(\varepsilon^{J'}_J)^*(\varepsilon^{J''}_{J'})^*(\varepsilon^{J''}_{J'})_*(\varepsilon^{J'}_{J_1})_*(\varepsilon^{J'}_{J_1})^*(\varepsilon^{J''}_{J'})_*(\varepsilon^{J'}_J)_* a=$$
$$=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_1})_*(\varepsilon^{J''}_{J_1})^*(\varepsilon^{J''}_J)_* a.$$

Note here that $J\cap J'=J$, $J_2\cap J'=J_1$, $J_1\cap J_2=J_1$. Using these correspondences, we have

$$(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_J)_* a=$$
$$=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_J)_*(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_J)_* a=$$
$$=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_{J\cap J'})_*(\varepsilon^{J''}_{J\cap J'})^*(\varepsilon^{J''}_J)_* a=\quad .$$
$$=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_{J'})_*(\varepsilon^{J''}_{J'})^*(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_J)_* a=$$
$$=(\varepsilon^{J''}_J)^*(\varepsilon^{J''}_{J_2})_*(\varepsilon^{J''}_{J_2})^*(\varepsilon^{J''}_{J'})_*(\varepsilon^{J''}_{J'})^*(\varepsilon^{J''}_J)_* a=$$

$$= (\varepsilon_J^{J''})^*(\varepsilon_{J_2}^{J''})_*(\varepsilon_{J_2}^{J''})^*(\varepsilon_{J_2\cap J'}^{J''})_*(\varepsilon_{J_2\cap J'}^{J''})^*(\varepsilon_J^{J''})_*a =$$
$$= (\varepsilon_J^{J''})^*(\varepsilon_{J_1}^{J''})_*(\varepsilon_{J_1}^{J''})^*(\varepsilon_J^{J''})_*a$$

Note that we used proposition 1.5 here.

CASE II. Let $n_{J'}: J' \to \Gamma$, $n_{J''}: J'' \to \Gamma$ be any objects, such that $J \subset J'$, $J \subset J''$. Denote $I = J' \cup J''$. Using CASE I, we obtain

$$(\varepsilon_J^{J'})^*(\varepsilon_{J_1}^{J'})_*(\varepsilon_{J_1}^{J'})^*(\varepsilon_J^{J'})_*a = (\varepsilon_J^{I})^*(\varepsilon_{J_1'}^{I})_*(\varepsilon_{J_1'}^{I})^*(\varepsilon_J^{I})_*a,$$
$$(\varepsilon_J^{J''})^*(\varepsilon_{J_2}^{J''})_*(\varepsilon_{J_2}^{J''})^*(\varepsilon_J^{J''})_*a = (\varepsilon_J^{I})^*(\varepsilon_{J_1'}^{I})_*(\varepsilon_{J_1'}^{I})^*(\varepsilon_J^{I})_*a,$$

$a \in R(J)$, $J_1' = I \setminus J$ •

Now we are ready to complete the proof that the axiom Q2 holds for the case $J_1 \cup J_2 = J$. We will prove it for the case $J_1 \cap J_2 = \varnothing$, because it is easy to understand that the general case follows from all those considered.

PROPOSITION 2.7. Assume that for any objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ such that $J_1 \subset J$, $J_2 \subset J$ takes place $J_1 \cap J_2 = \varnothing$ and $J_1 \cup J_2 = J$. Then for every $a \in R(J)$

$$\exists_J^{J_1}\exists_J^{J_2}a = \exists_J^J a .$$

PROOF. Assume that $n_I: I \to \Gamma$ be an object such that $J \subset I$. Then

$$\exists_J^{J_1}\exists_J^{J_2}a = (\varepsilon_{J_3}^{J})_*(\varepsilon_{J_3}^{J})^*(\varepsilon_{J_4}^{J})_*(\varepsilon_{J_4}^{J})^*a,$$

where $J_3 = J \setminus J_3$, $J_4 = J \setminus J_2$. We obtain

$$\exists_J^{J_1}\exists_J^{J_2}a =$$
$$= (\varepsilon_J^{I})^*(\varepsilon_J^{I})_*(\varepsilon_{J_3}^{J})_*(\varepsilon_{J_3}^{J})^*(\varepsilon_J^{I})^*(\varepsilon_J^{I})_*(\varepsilon_{J_4}^{J})_*(\varepsilon_{J_4}^{J})^*(\varepsilon_J^{I})^*(\varepsilon_J^{I})_*a = .$$
$$= (\varepsilon_J^{I})^*(\varepsilon_{J_3}^{I})_*(\varepsilon_{J_3}^{I})^*(\varepsilon_{J_4}^{I})_*(\varepsilon_{J_4}^{I})^*(\varepsilon_J^{I})_*a$$

On the other hand, using the propositions 1.5 and 1.6, we get

$$\exists_J^J a = (\varepsilon_J^{I})^*(\varepsilon_0^{I})_*(\varepsilon_0^{I})^*(\varepsilon_J^{I})_*a = (\varepsilon_J^{I})^*(\varepsilon_{J_0'\cap J_0''}^{I})_*(\varepsilon_{J_0'\cap J_0''}^{I})^*(\varepsilon_J^{I})_*a,$$

where $J_0 = I \setminus J$, $J_0' = I \setminus J_1$, $J_0'' = I \setminus J_2$. Then

$$(\varepsilon_J^{I})^*(\varepsilon_{J_0'\cap J_0''}^{I})_*(\varepsilon_{J_0'\cap J_0''}^{I})^*(\varepsilon_J^{I})_*a =$$
$$= (\varepsilon_J^{I})^*(\varepsilon_{J_0'}^{I})_*(\varepsilon_{J_0'}^{I})^*(\varepsilon_{J_0''}^{I})_*(\varepsilon_{J_0''}^{I})^*(\varepsilon_J^{I})_*a =$$
$$= (\varepsilon_J^{I})^*(\varepsilon_J^{I})_*(\varepsilon_J^{I})^*(\varepsilon_{J_0'}^{I})_*(\varepsilon_{J_0'}^{I})^*(\varepsilon_{J_0''}^{I})_*(\varepsilon_{J_0''}^{I})^*(\varepsilon_J^{I})_*(\varepsilon_J^{I})^*(\varepsilon_J^{I})_*a =$$
$$= (\varepsilon_J^{I})^*(\varepsilon_{J\cap J_0'}^{I})_*(\varepsilon_{J\cap J_0'}^{I})^*(\varepsilon_{J'\cap J_0''}^{I})_*(\varepsilon_{J'\cap J_0''}^{I})^*(\varepsilon_J^{I})_*a =$$
$$= (\varepsilon_J^{I})^*(\varepsilon_{J_3}^{I})_*(\varepsilon_{J_3}^{I})^*(\varepsilon_{J_4}^{I})_*(\varepsilon_{J_4}^{I})^*(\varepsilon_J^{I})_*a$$

because of $J \cap J_0' = J_3$, $J \cap J_0'' = J_4$ •

So, we completed the proof of the theorem 1.1.

1.1 Operation of Halmos Algebras without equality

In this section we will consider the category $\mathfrak{R}_2$ of the Relational Algebras over the scheme-category K_2, in which only injective morphisms between the objects were defined. Fix an arbitrary Relational Algebra $R \in Ob\mathfrak{R}_2$.

Since the category K_1 is a subcategory of the category K_2 theorem 2.1 holds for any RA from the category $\mathfrak{R}_2$. Therefore any BA *R(J)* of RA R is a Quantifier Algebra. Then for every object $n: J \to \Gamma$ of the scheme-category K_2 denote by S_J the semigroup of all the transformations (morphisms) s of J, which fixed the sorts of elements from J. Then, according to the axioms of a Relational Algebra, for every $s \in S_J$ we have an endomorphism of the Boolean Algebra

$$s_*: R(J) \to R(J).$$

Thus we defined an action of an element $s \in S_J$ as an endomorphism of the Boolean Algebra *R(J)* and according to the axioms of RA the unit of S_J acts as the identity endomorphism of *R(J)*.

It is not hard to understand that any Boolean Algebra *R(J)* here can be presented as a Transformational Algebra. For any morphism $s \in S_J$, we have defined a representation s as an endomorphism s_* of the Boolean Algebra *R(J)*.

THEOREM 2.2. Let R be a Relational Algebra over the scheme-category K_2. Then every Boolean Algebra *R(J)* of the Relational Algebra R may be transformed to a Halmos Algebra (without an identity) over the scheme $n: J \to \Gamma$, i.e. the following axioms (in this case) hold:

1. $s_{1*}\exists_J^{J_1} a = s_{2*}\exists_J^{J_1} a,\ J_1 \subseteq J,\ s_1, s_2 \in S_J$ and $s_1\alpha = s_2\alpha$ for all $\alpha \in J \setminus J_1,\ \ a \in R(J)$;

2. $\exists_J^{J_1} s_* a = s_* \exists_J^{s^{-1}J_1} a,\ J_1 \subseteq J,\ s \in S_J,\ s^{-1}J_1 = \{\alpha | s\alpha \in J_1\}$ and s acts injectivly.

We will divide the proof of the theorem into several lemmas and propositions.

Note here that for the case $J_1 = J$ the proof of item 1 of theorem 2.2 is trivial, in the same time the second axiom of the theorem for the same case will look like

$$\exists_J^J s_* a = s_* \exists_J^J a$$

and we will prove it later.

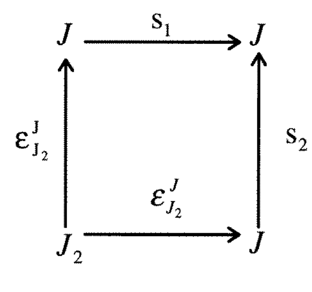
PROPOSITION 2.8. Let $J_1 \subseteq J,\ s_1, s_2 \in S_J$ and $s_1\alpha = s_2\alpha$ for all $\alpha \in J \setminus J_1$. Then for any $a \in R(J)$

$$s_{1*}\exists_J^{J_1} a = s_{2*}\exists_J^{J_1} a.$$

PROOF. We have

$$s_{1*}\exists_J^{J_1} a = s_{1*}(\varepsilon_{J_2}^J)_*(\varepsilon_{J_2}^J)^* a = (s_1 \cdot \varepsilon_{J_2}^J)_*(\varepsilon_{J_2}^J)^* a,\ J_2 = J \setminus J_1.$$

Let us examine the following diagram:



It is clear that it is commutative, i.e.

$$s_1 \cdot \varepsilon_{J_2}^J \alpha = s_1 \cdot \varepsilon_{J_2}^J \alpha, \alpha \in J_2..$$

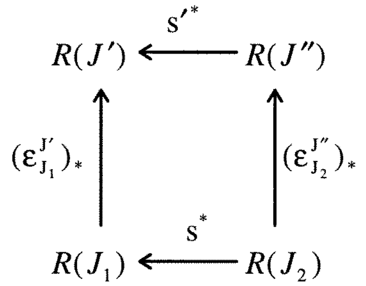
Therefore

$$s_1 \cdot \exists_J^{J_1} a = (s_1 \cdot \varepsilon_{J_2}^{J})_* (\varepsilon_{J_2}^{J})^* a = (s_2 \cdot \varepsilon_{J_2}^{J})_* (\varepsilon_{J_2}^{J})^* a =$$
$$= s_{2*} (\varepsilon_{J_2}^{J})_* (\varepsilon_{J_2}^{J})^* a = s_2 \cdot \exists_J^{J_1} a \bullet$$

We begin the proof of item 2 of theorem 2.2. For this in particular, we need the following important lemma.

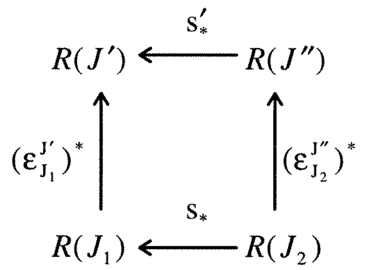
LEMMA 2.2. Let $s\colon J_1 \to J_2$ be an arbitrary morphism of any objects and assume that $n_{J_3}\colon J_3 \to \Gamma$ be an object such that $J_3 \cap J_1 = \varnothing$, $J_3 \cap J_2 = \varnothing$. Let us denote $J' = J_1 \cup J_3$, $J'' = J_2 \cup J_3$ and consider morphism $s'\colon J' \to J''$, where $s'\alpha = \alpha$ for $\alpha \in J_3$ and $s'\alpha = s\alpha$ for $\alpha \in J_1$. Then the following two commutative diagrams hold:

1.



i.e. $(\varepsilon_{J_1}^{J'})_* s^* a = s'^* (\varepsilon_{J_1}^{J''})_* a, a \in R(J_2)$.

2.



i.e. $(\varepsilon_{J_1}^{J''})^* s'_* b = s_* (\varepsilon_{J_2}^{J'})^* b, b \in R(J')$.

PROOF. 1. We can write

$$(\varepsilon_{J_1}^{J'})_* s^* a = s'^* (\varepsilon_{J_1}^{J''})_* s^* a \cap (\varepsilon_{J_3}^{J'}) 1_{R(J_3)}.$$

Using the fifth axiom of a Relational Algebra we get

$$(\varepsilon_{J_1}^{J'})_* s^* a = s'^* (\varepsilon_{J_1}^{J''})_* s^* a \cap (\varepsilon_{J_3}^{J'}) 1_{R(J_3)} = s'^* [(\varepsilon_{J_2}^{J''})_* a \cap (\varepsilon_{J_3}^{J''}) 1_{R(J_3)}] =$$
$$= s'^* (\varepsilon_{J_2}^{J''})_* a$$

Thus diagram 1 holds.

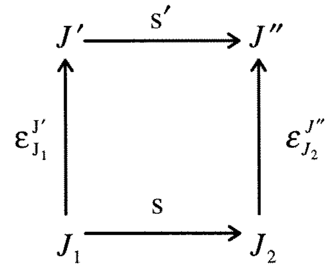
2. We want to show first, that

$$(\varepsilon_{J_1}^{J''})^* s'_* b \subset s_* (\varepsilon_{J_2}^{J'})^* b, b \in R(J').$$

We have

$$(\varepsilon_{J_1}^{J''})^* s'_* b \subset (\varepsilon_{J_1}^{J''})^* s'_* (\varepsilon_{J_1}^{J'})_* (\varepsilon_{J_1}^{J'})^* b = (\varepsilon_{J_1}^{J''})^* (s' \cdot \varepsilon_{J_1}^{J'})_* (\varepsilon_{J_1}^{J'})^* b.$$

Using commutativity of the following simple diagram



i.e. using the correspondence $\varepsilon_{J_2}^{J''} s\alpha = s'\varepsilon_{J_1}^{J'}\alpha, \alpha \in J_1$, we get

$$(\varepsilon_{J_1}^{J''})^* (s^* \cdot \varepsilon_{J_1}^{J'})_* (\varepsilon_{J_1}^{J'})^* b = (\varepsilon_{J_1}^{J''})^* (\varepsilon_{J_2}^{J''} \cdot s)_* (\varepsilon_{J_1}^{J'})^* b =$$
$$= (\varepsilon_{J_1}^{J''})^* (\varepsilon_{J_2}^{J''})_* s_* (\varepsilon_{J_1}^{J'})^* b =$$
$$= s_* (\varepsilon_{J_1}^{J'})^* b$$

Now we have to check that the correspondence

$$(\varepsilon_{J_2}^{J''})^* s'_* b \supset s_* (\varepsilon_{J_1}^{J'})^* b$$

for any $b \neq 0$ holds. Denote $t = s_* (\varepsilon_{J_1}^{J'})^* b \setminus (\varepsilon_{J_2}^{J''})^* s'_* b$. So we have

(A) $t \subseteq s_* (\varepsilon_{J_1}^{J''})^* b,$

(B) $t \cap (\varepsilon_{J_2}^{J''})^* s'_* b = 0.$

Assume $t \neq 0$. Let us suppose that *t=0*. According to items 3 and 4 of proposition 2.1 we can write for the expression (A)

$$t \subseteq s_* (\varepsilon_{J_1}^{J''})^* b \Rightarrow s^* t \subseteq (\varepsilon_{J_1}^{J''})^* b.$$

Using the item 5 of proposition 1.1, we get for item (B)

$$t \cap (\varepsilon_{J_2}^{J''})^* s'_* b = 0 \Rightarrow (\varepsilon_{J_2}^{J''})_* t \cap s'_* b = 0 \Rightarrow s'^* (\varepsilon_{J_2}^{J''})_* t \cap b = 0.$$

By the hypothesis $t \neq 0$, we want to check if $s'^* (\varepsilon_{J_2}^{J''})_* t \neq 0$ and $s^* t \neq 0$ hold.
Indeed, assume $t \neq 0$ and let us propose $(\varepsilon_{J_2}^{J''})_* t = 0.$ It implies $(\varepsilon_{J_2}^{J''})^* (\varepsilon_{J_2}^{J''})_* t = (\varepsilon_{J_2}^{J''})_* t$, i.e. *t=0*. Assume, further, that $(\varepsilon_{J_2}^{J''})_* t \neq 0$ and let $s'^* (\varepsilon_{J_2}^{J''})_* t = 0$. Then we get

$$s'^* (\varepsilon_{J_2}^{J''})_* t = 0 \Rightarrow s'_* s'^* (\varepsilon_{J_2}^{J''})_* t = s'_* 0 \Rightarrow s'_* s'^* (\varepsilon_{J_2}^{J''})_* t = 0.$$

However we have contradiction, because

$$(\varepsilon_{J_2}^{J''})_* t \subset s'_* s'^* (\varepsilon_{J_2}^{J''})_* t \text{ and } s'_* s'^* (\varepsilon_{J_2}^{J''})_* t = 0.$$

So we checked that since $t \neq 0$, it follows that $s'^* (\varepsilon_{J}^{J''})_* t \neq 0$. The condition

$$t \neq 0 \Rightarrow s^* t \neq 0$$

can be proved similarly.

So, we have $s^* t \subseteq (\varepsilon_{J_1}^{J'})^* b$ and $s'^* (\varepsilon_{J_2}^{J''})_* t \cap b = 0$, but using item 1 of the present lemma we get

$$s'^* (\varepsilon_{J_2}^{J''})_* t = (\varepsilon_{J_1}^{J'})_* s^* t.$$

Hence

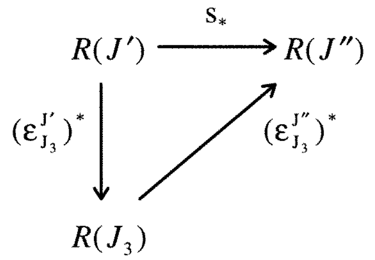
$$s'^* (\varepsilon_{J_2}^{J''})_* t \cap b = 0 \Rightarrow (\varepsilon_{J_1}^{J'})_* s^* t \cap b = 0 \Rightarrow s^* t \cap (\varepsilon_{J_1}^{J'})^* b = 0,$$

i.e. $s^* t = 0$ (because of $(\varepsilon_{J_1}^{J'})^* b \neq 0$). This is a contradiction, from which it follows that $t{=}0$. Thus, we have

$$(\varepsilon_{J_1}^{J''})^* s'_* b \subset s_* (\varepsilon_{J_2}^{J'})^* b \text{ and } s_* (\varepsilon_{J_1}^{J'})^* b \setminus (\varepsilon_{J_2}^{J''})^* s'_* b = 0$$

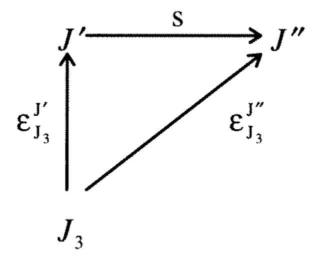
therefore $(\varepsilon_{J_1}^{J''})^* s'_* b = s_* (\varepsilon_{J_2}^{J'})^* b$ •

LEMMA 2.3. Let $s_1 : J_1 \to J_2$ be an injective morphism and assume $n_{J_3} : J_3 \to \Gamma$ be an object such that $J_3 \cap J_1 = \varnothing$, $J_3 \cap J_2 = \varnothing$. Let us denote $J' = J_1 \cup J_3$, $J'' = J_2 \cup J_3$ and consider the morphism $s: J' \to J''$, where $s\alpha = \alpha$ for $\alpha \in J_3$ and $s\alpha = s_1\alpha$ for $\alpha \in J_1$. Then the following commutative diagram holds:



i.e. $(\varepsilon_{J_3}^{J''})^* s_* a = (\varepsilon_{J_3}^{J'})^* a, a \in R(J')$.

PROOF. It is easy to see that the following commutative diagram



takes place, i.e. $s \cdot \varepsilon_{J_3}^{J'} \alpha = \varepsilon_{J_3}^{J''} \alpha, \alpha \in J_3$. It implies $(s \cdot \varepsilon_{J_3}^{J'})^* a = (\varepsilon_{J_3}^{J''})^* a$ and then $(\varepsilon_{J_3}^{J'})^* s^* a = (\varepsilon_{J_3}^{J''})^* a, a \in R(J')$.. Since s is an injective morphism we have $s^* s_* a = a, a \in R(J')$. Therefore

$$(\varepsilon_{J_3}^{J'})^* a = (\varepsilon_{J_3}^{J'})^* s^* s_* a = (\varepsilon_{J_3}^{J''})^* s_* a \bullet$$

Now we can prove item 2 of theorem 1.3. First we will make it for the case when J_1 is a proper subset of J.

PROPOSITION 2.9. Assume that J_1 is a proper subset of J where $n_J : J \to \Gamma$ is an object, $s \in S_J$ and let s act injectively on $s^{-1} J_1 = \{\alpha | s\alpha \in J_1\}$. Then for $\forall a \in R(J)$

$$\exists_J^{J_1} s_* a = s_* \exists_J^{s^{-1} J_1} a.$$

PROOF. We have

$$\exists_J^{J_1} s_* a = (\varepsilon_{J_2}^{J})_* (\varepsilon_{J_2}^{J})^* s_* a, J_2 = J \setminus J_1;$$

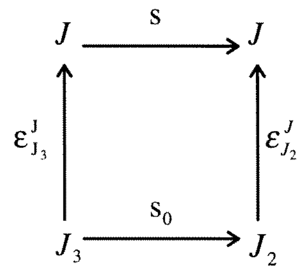
$$s_* \exists_J^{s^{-1} J_1} a = s_* (\varepsilon_{J_3}^{J})_* (\varepsilon_{J_3}^{J})^* a, J_3 = J \setminus s^{-1} J_1.$$

So, we must prove that

$$(\varepsilon_{J_2}^{J})_* (\varepsilon_{J_2}^{J})^* s_* a = s_* (\varepsilon_{J_3}^{J})_* (\varepsilon_{J_3}^{J})^* a.$$

Since s injectively acts on $s^{-1} J_1$ then $s_0 : J_3 \to J_2$ is an injective morphism, where s_0 is a restriction s on J_3.

Consider the following diagram :



Since it is commutative, i.e. $s \cdot \varepsilon^J_{J_3}\alpha = \varepsilon^J_{J_2} \cdot s_0\alpha, \quad \alpha \in J_{3,}$, we get

$$s_*(\varepsilon^J_{J_3})_* a = (\varepsilon^J_{J_2})_* s_{0^*} a, a \in R(J_3).$$

Returning to our correspondence we have

$$s_*(\varepsilon^J_{J_3})_*(\varepsilon^J_{J_3})^* a = (\varepsilon^J_{J_2})_* s_{0^*}(\varepsilon^J_{J_3})^* a.$$

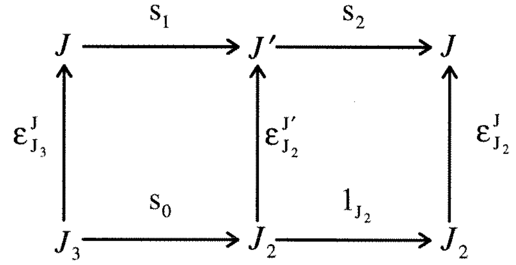
Hence it is necessary to check that

$$(\varepsilon^J_{J_2})_* s_{0^*}(\varepsilon^J_{J_3})^* a = (\varepsilon^J_{J_2})_*(\varepsilon^J_{J_2})^* s_* a.$$

Taking into account that $(\varepsilon^J_{J_2})_*$ is a monomorphism of the corresponding Boolean Algebras we have to prove that

$$s_{0^*}(\varepsilon^J_{J_3})^* a = (\varepsilon^J_{J_2})^* s_* a.$$

Now let us consider the diagram

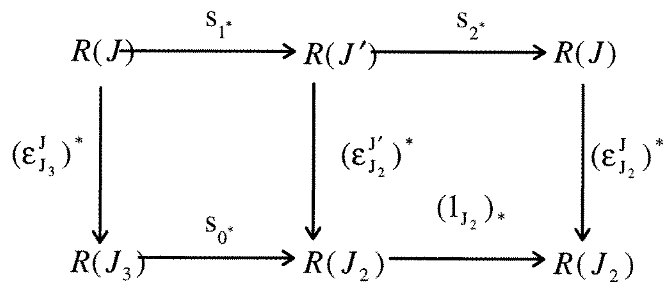


where $J' = J_2 \cup J_5$, $n_{J_5}: J_5 \to \Gamma$ is an arbitrary object such that there exists a bijective morphism $\sigma: J_1 \to J_5$. If we define s_1 and s_2 by the rules

$$s_1\alpha = \begin{cases} s_0\alpha, \alpha \in J_{3,} \\ \sigma\alpha, \alpha \in s^{-1}J_1; \end{cases}$$

$$s_2\alpha = \begin{cases} s\sigma^{-1}\alpha, \alpha \in J_{5,} \\ \alpha, \alpha \in J_2; \end{cases}$$

then it is simple to see that the above diagram is commutative. It naturally implies the commutativity of the following diagram:



Indeed, the left part of it is commutative by lemma 2.2 (item 2) and right one by lemma 2.3. So we have

$$(\varepsilon^J_{J_2})^* s_{2^*} s_{1^*} a = (1_{J_2})_* s_{0^*}(\varepsilon^J_{J_3})^* a$$

for $a \in R(J)$. Thus, we get

$$(\varepsilon^{J}_{J_2})^* s_{2^*} s_{1^*} a = (\varepsilon^{J}_{J_2})^* s_* a = s_{0^*} (\varepsilon^{J}_{J_3})^* a$$

since $s_{2^*} s_{1^*} a = (s_2 s_1)_* a = s_* a$ •

Consider the last case $J_1 = J$ for axiom 2 from theorem 2.2.

PROPOSITION 2.10. Assume $s: J \to J$ is a bijective morphism for any object $n_J : J \to \Gamma$. Then

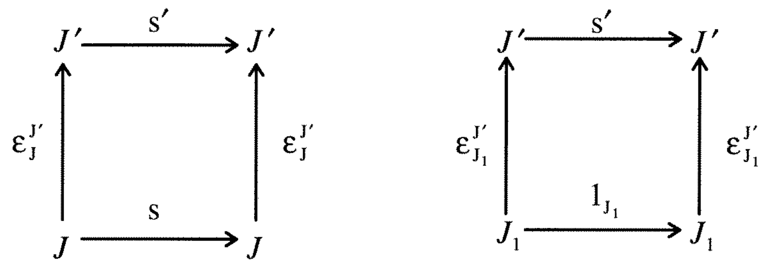
$$\exists^J_J s_* a = s_* \exists^J_J a, a \in R(J).$$

PROOF. Let $n_{J'} : J' \to \Gamma$ be an object such that J is a proper subset of J'. By definition 2.2 we have

$$s_* \exists^J_J a = s_* (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* a,$$

$$\exists^J_J s_* a = (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* s_* a, J_1 = J' \setminus J.$$

Take into consideration the following commutative diagrams



where $s': J' \to J'$ is a morphism which we define according to the rule $s'\alpha = s\alpha$ if $a \in J$ and $s'\alpha = \alpha$ for $\alpha \in J' \setminus J$. It is clear that both these diagrams are commutative. In other words, we can write for the first diagram and for the second one, respectively $\varepsilon^{J'}_J s\alpha = s'\varepsilon^{J'}_J \alpha, \alpha \in J$; $\varepsilon^{J'}_{J_1} 1_{J_1} \alpha = s'\varepsilon^{J'}_{J_1} \alpha, \alpha \in J_1$. Using these expressions we get

$$\exists^J_J s_* a = (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J s)_* a = (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (s'\varepsilon^{J'}_J)_* a =$$
$$= (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* s'_* (\varepsilon^{J'}_J)_* a.$$

Since $(\varepsilon^{J'}_{J_1})^* s'_* b = (\varepsilon^{J'}_{J_1})^* b, b \in R(J')$ (by lemma 2.2) we have for $a \in R(J)$

$$\exists^J_J s_* a = \exists^J_J a$$

On the other hand ,

$$s_* \exists^J_J a = s_* (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* =$$
$$= (\varepsilon^{J'}_J)^* s'_* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* a =$$
$$= (\varepsilon^{J'}_J)^* (s'\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* a = (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1} 1_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* a =$$
$$= (\varepsilon^{J'}_J)^* (\varepsilon^{J'}_{J_1})_* (\varepsilon^{J'}_{J_1})^* (\varepsilon^{J'}_J)_* a = \exists^J_J a$$ •

Thus the proof of theorem 2.2 is complete.

2.2 Operations of Halmos Algebra with equality

Now we will study the category $\mathfrak{R}$ of the Relational Algebras over the scheme-category K , in which we will consider arbitrary morphisms. It is clear that theorems 2.1 and 2.2 hold here, moreover, theorem 2.2 holds for any BA *R(J)* of RA from the category $\mathfrak{R}$, it means that items 1 and 2 of theorem 2.2 take places for any corresponding morphism (not only injective).

Now we want to define an identity in any BA *R(J)* of a Relational Algebra *R* from the category $\mathfrak{R}$.

DEFINITION 2.3. For any elements $\alpha_1, \alpha_2 \in J$ of the same sort, i.e. such that $n_J(\alpha_1) = n_J(\alpha_2)$, we set

$$d_{R(J)}(\alpha_1, \alpha_2) = (s_{\alpha_1}^{\alpha_2})^* 1_{R(J)}$$

where $s_{\alpha_1}^{\alpha_2} : J \to J$ is the replacement of *J* such that $s_{\alpha_1}^{\alpha_2}(\alpha_1) = \alpha_2$ and $s_{\alpha_1}^{\alpha_2}$ acts identically for the rest of $\alpha \in J$, $1_{R(J)}$ is the unit of BA *R(J)*.

Note that we could not define an identity in any BA *R(J)* of RA because there exist only injective morphisms in category K_2 , and it is clear that no replacement is an injection.

THEOREM 2.3. The axioms of an identity hold in every Boolean Algebra *R(J)* of a Relational Algebra *R:*

1. $s_*(s_{\alpha_1}^{\alpha_2})^* 1 = (s_{s\alpha_1}^{s\alpha_2})^* 1,\ s \in S_J$,
2. $(s_\alpha^\alpha)^* 1 = 1$,
3. $a \cap (s_{\alpha_1}^{\alpha_2})^* 1 \subseteq (s_{\alpha_1}^{\alpha_2})^* a$.

It is obvious that the second axiom holds. As above we divide the proof for several lemmas and propositions.

The following lemma allows us to understand the simple fact that for any BA *R(J)* the element $(s_{\alpha_1}^{\alpha_2})^* 1_{R(J)}$ is supported by the set $\{\alpha_1, \alpha_2\}$ (more precisely, the set $\{\alpha_1, \alpha_2\}$ is a minimal support of the element $(s_{\alpha_1}^{\alpha_2})^* 1_{R(J)}$) .

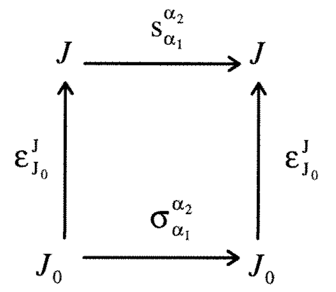
LEMMA 2.4. Assume that $n_J : J \to \Gamma$ is an object of the scheme K, $\alpha_1, \alpha_2 \in J$ and *R(J)* is the corresponding Boolean Algebra. Denote $J_0 = \{\alpha_1, \alpha_2\}$. Then

$$(s_{\alpha_1}^{\alpha_2})^* 1 = (\varepsilon_{J_0}^J)_* (\varepsilon_{J_0}^J)^* (s_{\alpha_1}^{\alpha_2})^* 1.$$

PROOF. We have

$$(\varepsilon_{J_0}^J)_* (\varepsilon_{J_0}^J)^* (s_{\alpha_1}^{\alpha_2})^* 1 = (\varepsilon_{J_0}^J)_* (s_{\alpha_1}^{\alpha_2} \varepsilon_J^J)^* 1.$$

Evidently the necessary diagram



is commutative, i.e. for $\alpha \in J_0$

$$\varepsilon^J_{J_0}\sigma^{\alpha_2}_{\alpha_1}\alpha = s^{\alpha_2}_{\alpha_1}\varepsilon^J_{J_0}\alpha,$$

where $\sigma^{\alpha_2}_{\alpha_1}: J' \to J'$ is the corresponding replacement. Using this we get

$$(\varepsilon^J_{J_0})_*(s^{\alpha_2}_{\alpha_1}\varepsilon^J_J)^*1 = (\varepsilon^J_{J_0})_*(\varepsilon^J_{J_0}\sigma^{\alpha_2}_{\alpha_1}\alpha)^*1 = (\varepsilon^J_{J_0})_*(\sigma^{\alpha_2}_{\alpha_1})^*(\varepsilon^J_{J_0})^*1 =$$
$$= (\varepsilon^J_{J_0})_*(\sigma^{\alpha_2}_{\alpha_1})^*(\varepsilon^J_{J_0})^*1 \cap (\varepsilon^J_{J_2})_*(\varepsilon^J_{J_2})^*1,$$

where $J_2 = J \setminus J_0$. Now we are ready to use the fifth axiom of a Relational Algebra. So we can write

$$(\varepsilon^J_{J_0})_*(\sigma^{\alpha_2}_{\alpha_1})^*(\varepsilon^J_{J_0})^*1 \cap (\varepsilon^J_{J_2})_*(\varepsilon^J_{J_2})^*1 =$$
$$(s^{\alpha_2}_{\alpha_1})^*[(\varepsilon^J_{J_0})_*(\varepsilon^J_{J_0})^*1 \cap (\varepsilon^J_{J_2})_*(\varepsilon^J_{J_2})^*1] = (s^{\alpha_2}_{\alpha_1})^*1 \bullet$$

LEMMA 2.5. Assume that $s_1, s_2 \in S_J$ are two elements such that $s_1\alpha_1 = s_2\alpha_1$, $s_1\alpha_2 = s_2\alpha_2$, $\alpha_1, \alpha_2 \in J$, and let action of s_1, s_2 be arbitrary for the rest of the elements from *J*. Then

$$s_{1^*}(s^{\alpha_2}_{\alpha_1})^*1 = s_{2^*}(s^{\alpha_2}_{\alpha_1})^*1.$$

PROOF. Using lemma 2.4 and item 1 of theorem 2.2 we have

$$s_{1^*}(s^{\alpha_2}_{\alpha_1})^*1 = s_{1^*}(\varepsilon^J_{J_0})_*(\varepsilon^J_{J_0})^*(s^{\alpha_2}_{\alpha_1})^*1 = s_{1^*}(\varepsilon^J_{J_0})_*(\varepsilon^J_{J_0})^*(s^{\alpha_2}_{\alpha_1})^*1 =$$
$$= s_{2^*}(s^{\alpha_2}_{\alpha_1})^*1 \bullet$$

PROPOSITION 2.11. The first axiom of theorem 2.3 holds, i.e. for any object $n_J: J \to \Gamma$ of the category K the following holds

$$s_*(s^{\alpha_2}_{\alpha_1})^*1 = (s^{s\alpha_2}_{s\alpha_1})^*1, \quad s \in S_J.$$

PROOF. Consider any BA *R(J)* and assume $\alpha_1, \alpha_2 \in J$, $s \in S_J$. Let us examine two cases.

CASE I. Assume $s\alpha_1 \neq s\alpha_2$. Denote $s\alpha_1 = \beta$, $s\alpha_2 = \gamma$.. Construct the morphism $s_0 \in S$ by the rule $s_0\alpha_1 = s\alpha_2 = \beta$, $s_0\alpha_2 = s\alpha_2 = \gamma$, $s_0\beta = \alpha_1$, $s_0\gamma = \alpha_2$ and let the action s_0 for the rest of the elements from *J* be identical. It is clear that s_0 is a bijection. Assume also $\alpha_1 \neq \beta \neq \alpha_2 \neq \gamma$. Note that $s_0 s_0 = 1_J$. Then by lemma 2.5

$$s_*(s^{\alpha_2}_{\alpha_1})^*1 = s_{0^*}(s^{\alpha_2}_{\alpha_1})^*1.$$

We have

$$(s^{s\alpha_2}_{s\alpha_1})^*1 = (s^{\gamma}_{\beta})^*1 = (s_0 \cdot s^{\alpha_2}_{\alpha_1} \cdot s_0)^*1 = s_0{}^*(s^{\alpha_2}_{\alpha_1})^* s_0{}^*1 = s_0{}^*(s^{\alpha_2}_{\alpha_1})^* s_0{}^*(s_{0^*}1) =$$
$$= s_0{}^*(s_{0^*}s_{0^*})(s^{\alpha_2}_{\alpha_1})^*(s_0{}^*s_{0^*})1 = (s_0{}^*s_{0^*})s_{0^*}(s^{\alpha_2}_{\alpha_1})^*1 = s_{0^*}(s^{\alpha_2}_{\alpha_1})^*1 =$$
$$= s_*(s^{\alpha_2}_{\alpha_1})^*1.$$

Now we are going to show that the case when α_1 or α_2 are equal to β or γ, it reduces to the just considered CASE 1. Indeed, assume, for instance, $\alpha_2 = \beta$, i.e. in this case

$$(s^{s\alpha_2}_{s\alpha_1})^*1 = (s^{\gamma}_{\beta})^*1 = (s^{\gamma}_{\alpha})^*1.$$

So we have to prove that

$$s_*(s^{\alpha_2}_{\alpha_1})^*1 = (s^{\gamma}_{\alpha_2})^*1.$$

Assume, first, that *J* contains at least two different variables δ_1, δ_2 of the same kind as α_1, i.e. $\delta_1 \neq \delta_2 \neq \alpha_1 \neq \alpha_2$ and $n_J(\alpha_1) = n_J(\delta_1) = n_J(\delta_2)$. Let us construct $s_1, s_2 \in S$ by the rules $s_1\alpha_1 = \delta_1$, $s_1\alpha_2 = \delta_2$ and let s_1 act identically on the rest of the elements from *J;*

$s_2\delta_1=\alpha_2$, $s_2\delta_2=\gamma$ and $s_2\alpha=\alpha$, $\alpha\in J\setminus\{\delta_1,\delta_2\}$. According to lemma 2.5, we get

$$s_*(s_{\alpha_1}^{\alpha_2})^*1=s'_*(s_{\alpha_1}^{\alpha_2})^*1,$$

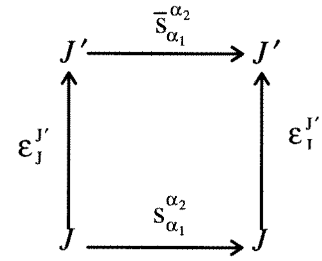
where $s'\in S_J$ and $s'\alpha_1=s\alpha_1$, $s'\alpha_2=s\alpha_2$ and $s'\alpha=\alpha$, $\alpha\in J\setminus\{\alpha_1,\alpha_2\}$. Using the above proved statement, we can write

$$s_*(s_{\alpha_1}^{\alpha_2})^*1=s'_*(s_{\alpha_1}^{\alpha_2})^*1=(s_1s_2)_*(s_{\alpha_1}^{\alpha_2})^*1=s_{2*}s_{1*}(s_{\alpha_1}^{\alpha_2})^*1=$$
$$=s_{2*}(s_{s_1\alpha_1}^{s_1\alpha_2})^*1=s_{2*}(s_{\delta_1}^{\delta_2})^*1=(s_{s_2\delta_1}^{s_2\delta_2})^*1=(s_{\alpha_2}^{\gamma})^*1.$$

Assume now that J does not contain two required variables. Then let $n_{J'}:J'\to\Gamma$ be any object such that the set J' contains required elements. We have

$$(s_{\alpha_1}^{\alpha_2})^*1=(s_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})^*(\varepsilon_J^{J'})_*1=(\varepsilon_J^{J'}\cdot s_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1.$$

It is easy to understand that the following diagram



where $\bar{s}_{\alpha_1}^{\alpha_2}\in S_{J'}$, is commutative, i.e.

$$\varepsilon_J^{J'}\cdot s_{\alpha_1}^{\alpha_2}\alpha=\bar{s}_{\alpha_1}^{\alpha_2}\cdot\varepsilon_J^{J'}\alpha,\alpha\in J.$$

We continue

$$(s_{\alpha_1}^{\alpha_2})^*1=(\varepsilon_J^{J'}\cdot s_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1=(\bar{s}_{\alpha_1}^{\alpha_2}\cdot\varepsilon_J^{J'}\alpha)^*(\varepsilon_J^{J'})_*1=$$
$$=(\varepsilon_J^{J'})^*(\bar{s}_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1,$$

and therefore

$$s_*(s_{\alpha_1}^{\alpha_2})^*1=s_*(\varepsilon_J^{J'})^*(\bar{s}_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1.$$

And now, according to lemma 2.2 (item 2) we have

$$s_*(\varepsilon_J^{J'})^*(\bar{s}_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1=(\varepsilon_J^{J'})^*s'_*(\bar{s}_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1.$$

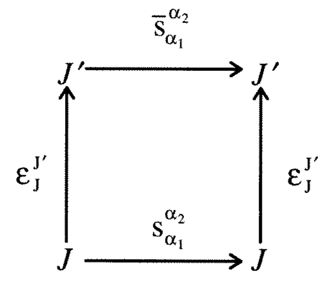
Here $s'\alpha=s\alpha,\alpha\in J$ and $s'\alpha=\alpha,\alpha\in J'\setminus J$. So we arrive at the case we have already studied. So

$$(\varepsilon_J^{J'})^*s'_*(\bar{s}_{\alpha_1}^{\alpha_2})^*(\varepsilon_J^{J'})_*1=(\varepsilon_J^{J'})^*(\bar{s}_{s'\alpha_1}^{s'\alpha_2})^*(\varepsilon_J^{J'})_*1,$$

and since $s'\alpha_1=s\alpha$, $s'\alpha_2=s\alpha_2$, we have

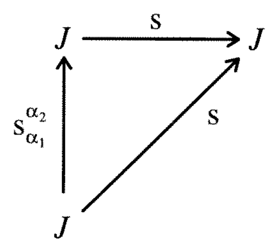
$$s_*(s_{\alpha_1}^{\alpha_2})^*1 = (\varepsilon_J^{J'})^*(\bar{s}_{s'\alpha_1}^{s'\alpha_2})^*(\varepsilon_J^{J'})_*1 = (\varepsilon_J^{J'})^*(\bar{s}_{s\alpha_1}^{s\alpha_2})^*(\varepsilon_J^{J'})_*1 =$$
$$= (\bar{s}_{s\alpha_1}^{s\alpha_2} \cdot \varepsilon_J^{J'})^*(\varepsilon_J^{J'})_*1 = (\varepsilon_J^{J'} \cdot s_{s\alpha_1}^{s\alpha_2})^*(\varepsilon_J^{J'})_*1 =$$
$$= (s_{s\alpha_1}^{s\alpha_2})^*(\varepsilon_J^{J'})^*(\varepsilon_J^{J'})_*1 = (s_{s\alpha_1}^{s\alpha_2})^*1.$$

We just used the following commutative diagram



i.e. $\bar{s}_{s\alpha_1}^{s\alpha_2} \cdot \varepsilon_J^{J'}\alpha = \varepsilon_J^{J'} \cdot s_{s\alpha_1}^{s\alpha_2}\alpha, \alpha \in J$.

CASE II. Assume $s\alpha_1 = s\alpha_2$. So, it is obvious that $(s_{s\alpha_1}^{s\alpha_2})^*1 = 1$. We have to check that $s_*(s_{\alpha_1}^{\alpha_2})^*1 = 1$. Since s_* is an endomorphism of the Boolean Algebra *R(J)*, then $s_*1 = 1$ and $s_*s^*1 = 1$ (the last follows from the axioms of RA). Since $s\alpha_1 = s\alpha_2$, it follows that the following diagram is commutative



i.e. $s\alpha = s \cdot s_{\alpha_1}^{\alpha_2}\alpha, \quad \alpha \in J$ .Using this get

$$s_*(s_{\alpha_1}^{\alpha_2})^*1 = (s \cdot s_{\alpha_1}^{\alpha_2})_*(s_{\alpha_1}^{\alpha_2})^*1 = s_*[(s_{\alpha_1}^{\alpha_2})_*(s_{\alpha_1}^{\alpha_2})^*]1 = 1 \bullet$$

Now we will prove that the third axiom of the theorem 2.3 holds.

PROPOSITION 2.12. For any object $n_J : J \to \Gamma$ and for every $a \in R(J)$

$$a \cap (s_{\alpha_1}^{\alpha_2})^*1 \subseteq (s_{\alpha_1}^{\alpha_2})_*a.$$

PROOF. We have

$$a \cap (s_{\alpha_1}^{\alpha_2})^*1 = a \cap (s_{\alpha_1}^{\alpha_2})^*1[(s_{\alpha_1}^{\alpha_2})_*a \cap (s_{\alpha_1}^{\alpha_2})_*\bar{a}] =$$
$$= a \cap [(s_{\alpha_1}^{\alpha_2})^*(s_{\alpha_1}^{\alpha_2})_*a \cup (s_{\alpha_1}^{\alpha_2})^*(s_{\alpha_1}^{\alpha_2})_*\bar{a}] =$$
$$= [a \cap (s_{\alpha_1}^{\alpha_2})^*(s_{\alpha_1}^{\alpha_2})_*a] \cup [a \cap (s_{\alpha_1}^{\alpha_2})^*(s_{\alpha_1}^{\alpha_2})_*\bar{a}].$$

Note that $a \cap (s_{\alpha_1}^{\alpha_2})^*(s_{\alpha_1}^{\alpha_2})_*\bar{a} \subseteq a \cap \bar{a} = 0$. Finally we get

$$a \cap (s_{\alpha_1}^{\alpha_2})^* 1 =$$

$$= [a \cap (s_{\alpha_1}^{\alpha_2})^* (s_{\alpha_1}^{\alpha_2})_* a] \cup [a \cap (s_{\alpha_1}^{\alpha_2})^* (s_{\alpha_1}^{\alpha_2})_* \overline{a}] \subseteq a \cap (s_{\alpha_1}^{\alpha_2})^* (s_{\alpha_1}^{\alpha_2})_* a =$$

$$= a \cap (s_{\alpha_1}^{\alpha_2})^* (s_{\alpha_1}^{\alpha_2} \cdot s_{\alpha_1}^{\alpha_2})_* a =$$

$$= a \cap (s_{\alpha_1}^{\alpha_2})^* (s_{\alpha_1}^{\alpha_2})_* [(s_{\alpha_1}^{\alpha_2})_* a] \subseteq a \cap (s_{\alpha_1}^{\alpha_2})_* a \subseteq (s_{\alpha_1}^{\alpha_2})_* a \bullet$$

CHAPTER 3

EQUIVALENCE OF THE CATEGORIES OF HALMOS ALGEBRAS AND RELATIONAL ALGEBRAS

First, we describe the construction of the direct spectrum of algebras and the direct limit of the direct spectrum [13]. A partially ordered set I is called directed if for any $\alpha,\beta\in I$ there exists an element $\gamma\in I$ such that $\alpha\leq\gamma$ and $\beta\leq\gamma$. A set of algebras M_α, enumerated by the elements of a directed set I is called a direct spectrum of algebras if for arbitrary elements $\alpha,\beta\in I$ there exists a monomorphism $\varphi: M_\alpha \to M_\beta$ such that $\alpha\leq\beta\leq\gamma$ implies

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta}\cdot\varphi_{\beta\gamma}\,.$$

We will say that a set $\boldsymbol{a}$ of the elements $a_\alpha \in M_\alpha$ is a thread if $a_\alpha \in \boldsymbol{a}$ implies that $\boldsymbol{a}$ contains also all the images $a_\alpha\varphi_{\alpha\beta}$ for $\beta>\alpha$ and all the preimages $a_\alpha\varphi_{\delta\alpha}^{-1}$, $\delta<\alpha$ for which such preimages exist.

It is easy to check that the set of all threads has the same structure as initial algebras. A set of all the threads is called the direct limit of the direct spectrum of algebras M_α.

Let us consider category-scheme K and let R be any Relational Algebra over K. For any objects $n_{J_1}: J_1 \to \Gamma$ and $n_{J_2}: J_2 \to \Gamma$ such that $J_1 \subseteq J_2$ consider the monomorphism of the Boolean Algebras

$$(\varepsilon_{J_1}^{J_2})_*: R(J_1) \to R(J_2)\,.$$

By this all the Boolean Algebras *R(J)* of the RA *R* are organized in a spectrum of BA-s. Denote by *H* a direct limit of the spectrum. *H* is a Boolean Algebra and for an object $n_J: J \to \Gamma$ contains subalgebra *H(J)* with the isomorphism

$$\mu(J): R(J) \to H(J)\,.$$

By the theorem 2.3 any Boolean Algebra *R(J)* from a Relational Algebra R may be considered as a Halmos Algebra over the scheme $n_J: J \to \Gamma$.

Therefore, the direct limit *H* of the direct spectrum of all the Halmos Algebras *R(J)* is also a Halmos Algebra. All the details are given in [18]. Denote this Halmos Algebra by *Hal(R)*.

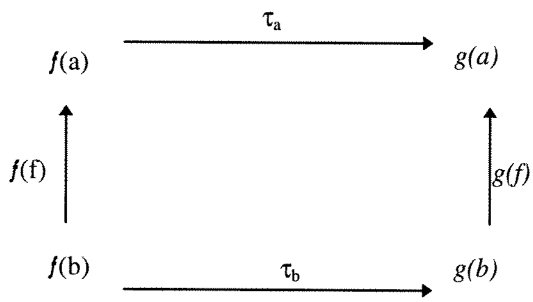
THEOREM. 3.1. The construction *Hal* is a functor from the category $\mathfrak{R}$ to the category $\aleph$.

PROOF is considered in [13] and [18].

Thus we have the following functors:

$$1.\ Hal: \Re \to \aleph,$$
$$2.\ Hal_1: \Re_1 \to \aleph_1$$
$$3.\ Hal_2: \Re_2 \to \aleph_2$$
$$4.\ Hal_3: \Re_3 \to \aleph_3\ .$$

Let L and D be two categories and let $f:L\text{->}D$ and $g:L\text{->}D$ be functors. Then natural transformation of a functor $f:L\text{->}D$ to a functor $g: L\text{->}D$ (see [13]) is a function τ, which corresponds D-morphism τ_α: $f(a)\text{->}g(a)$ to every object $a \in ObL$. Furthermore, for every L-morphism $f:a\text{->}b$, the following diagram



is commutative.

A natural transformation is called natural isomorphism if for every object a from L the morphism τ_α is an isomorphism (in the categorical terms).

So, the functor $f:L\text{->}D$ is called an equivalence of the categories L and D if there exists a functor $h:D\text{->}L$ together with natural isomorphisms $\tau: 1_L \text{->} h\cdot f$ and $\sigma: 1_D\text{->}f\cdot h$. We say that the categories L and D are equivalent, that is, $L \cong D$ if there exists an equivalence $f:L\text{->}D$.

The following theorem is the main result of the work.

THEOREM 3.1. The following categories are equivalent:

1. $\Re \cong \aleph$
2. $\Re_1 \cong \aleph_1$
3. $\Re_2 \cong \aleph_2$
4. $\Re_3 \cong \aleph_3$

PROOF. We have already built the necessary functors between the corresponding categories. To complete the proof of the theorem we examine only item 1 of the theorem, because items 2 - 4 will follow from it. Besides, we prove the necessary properties by dividing the proof into several theorems and propositions.

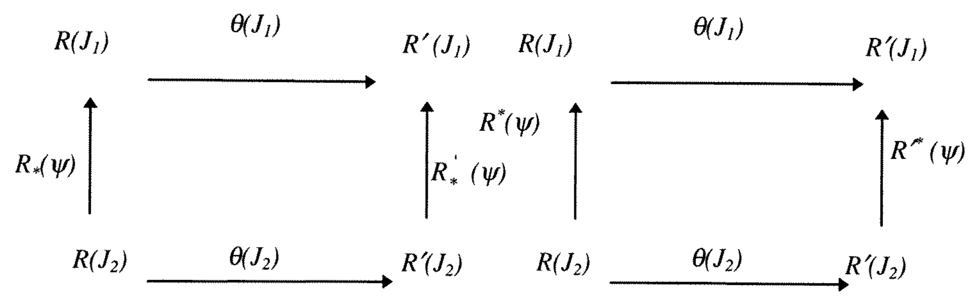
THEOREM 3.2. Suppose that $R \in Ob\Re$. Then there exists an isomorphism of the following RA-s

$$\chi:R\text{->}rel(HalR).$$

PROOF. Denote $HalR=H$, $relHal=relH=R'$. We have to construct an isomorphism $\chi:R\text{->}R'$. By theorem 1.1, $HalR$ is a Halmos Algebra and H is a collection of all $H(J)$ with the isomorphisms $\mu(J):R(J)\text{->}H(J)$. On the other hand, R' is a set of Boolean subalgebras $R'(J)$ in H and $R'(J)=\{h\in H \mid \exists^{\bar{J}} h=h\}$. By [PL], $R'(J)=H(J)$. So, for every finite set J we have an isomorphism $\mu(J):R(J)\text{->}R'(J)$ and for $\forall a\in R(J)$ we set

$$\chi(J)a=\mu(J)a.$$

Thus, for any J, we get an isomorphism $\chi(J)$ of the corresponding Boolean Algebras. We have to show now that by the definition of a homomorphism of the Relational Algebras, for arbitrary morphism $\psi:J_1\text{->}J_2$ the diagrams

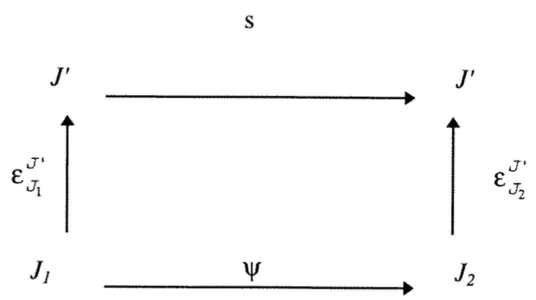


are commutative. We consider the case when $J_1\cap J_2\neq\varnothing$. It is clear that the general case follows from this.

Examine the first diagram. It is necessary to prove that for $\forall a\in R(J_1)$

$$R'_*(\psi)\mu(J_1)a=\mu(J_2)R_*(\psi)a.$$

Denote, $R_*(\psi)a=b$, $b\in R(J_2)$ and $\mu(J_2)b=h_2$. We have to check $R'_*(\psi)h_1=h_2$, where $h_1=\mu(J_1)a$. By the definition, $R'_*(\psi)h_1=\mu(J')R_*(s)c$, where $s\in S_{J'}$ is the corresponding element, $J'=J_1\cup J_2$, $c\in R(J')$ and $\mu(J')c=h_1$. Obvious that $c=R_*(\varepsilon_{J_1}^{J'})a$. Consider the diagram



The commutativity of the diagram yields for $\forall d \in R(J_1)$

$$R_*(s)R_*(\varepsilon_{J_1}^{J'})\,d = R_*(\varepsilon_{J_2}^{J'})\;R_*(\psi)d.$$

Using this we have

$$R'_*(\psi)h_1=\mu(J')R_*(s)c=\mu(J')R_*(s)R_*(\varepsilon_{J_2}^{J'})a=\mu(J')R_*(\varepsilon_{J_2}^{J'})\;R_*(\psi)a=$$
$$=\mu(J')R_*(\varepsilon_{J_2}^{J'})b.$$

Therefore, since $\mu(J_2)b=h_2$, we have

$$\mu(J')R_*(\varepsilon_{J_2}^{J'})b=h_2,$$

as desired.

Let us prove the commutativity of the second diagram. Assume $J_1=\{\alpha_1,\dots,\alpha_n\}$, $J_2=\{\beta_1,\dots,\beta_m\}$. We have to prove that for any $a\in R(J_2)$,

$$R'^*(\psi)\mu(J_2)a=\mu(J_1)R^*(\psi)a.$$

Assume $R^*(\psi)=b$, $b\in R(J_1)$. Denote $h_1=\mu(J_1)b$ and $h_2=\mu(J_2)a$. It is necessary to show that $R'^*(\psi)h_2=h_1$. By the definition,

$$R'^*(\psi)h_2=\exists^{J_2}(h_2\cap d(\alpha_1,\psi\alpha_1)\cap\ldots\cap d(\alpha_n,\psi\alpha_n)).$$

But we have defined an existential quantifier $\exists^{J_2}$ and $d(\alpha_i,\psi\alpha_i)$, i=1,...,n through the operations $_*$ and * in a suitable Boolean Algebra $R(J)$. We set $J'=J_1\cup J_2$ and let us write the expression $h_2\cap d(\alpha_1,\psi\alpha_1)\cap\ldots\cap d(\alpha_n,\psi\alpha_n)$ through the means of BA $R(J')$. This implies

$$h_2=\mu(J')R_*(\varepsilon_{J_2}^{J'})a.$$

By the corresponding definition (see [18]),

$$d(\alpha_i,\psi\alpha_i)=\mu(J')R^*(s_{\alpha_i}^{\psi\alpha_i})1,\quad i=1,\ldots,n,$$

where 1 is the unit of BA $R(J')$. So, we obtain

$$h_2\cap d(\alpha_i,\psi\alpha_i)\cap\ldots\cap d(\alpha_n,\psi\alpha_n)=\mu(J')R_*(\varepsilon_{J_2}^{J'})a\cap\mu(J')R^*(s_{\alpha_{1i}}^{\psi\alpha_1})1\cap\ldots\cap$$
$$\cap\mu(J')R^*(s_{\alpha_n}^{\psi\alpha_n})1=\mu(J')[R_*(\varepsilon_{J_2}^{J'})a\cap R^*(s_{\alpha_1}^{\psi\alpha_1})1\cap\ldots\cap\cap R^*(s_{\alpha_n}^{\psi\alpha_n})1].$$

Then

$$\exists^{J_2}(h_2\cap d(\alpha_1,\psi\alpha_1)\cap\ldots\cap d(\alpha_n,\psi\alpha_n))=$$
$$=\mu(J')[R_*(\varepsilon_{J_0}^{J^1})R^*(\varepsilon_{J_0}^{J^1})(R_*(\varepsilon_{J_2}^{J'})a\cap\cap R^*(s_{\alpha_1}^{\psi\alpha_1})1\cap\ldots\cap\cap R^*(s_{\alpha_n}^{\psi\alpha_n})1)],$$

where $J_0=J'\setminus J_2=J_1$. Then we get

$$\exists^{J_2}(h_2\cap d(\alpha_1,\psi\alpha_1)\cap\ldots\cap d(\alpha_n,\psi\alpha_n))=$$
$$=\mu(J')[R_*(\varepsilon_{J_0}^{J^1})R^*(\varepsilon_{J_0}^{J^1})(R_*(\varepsilon_{J_0}^{J^1})R^*(\psi)\alpha]=\mu(J')[R_*(\varepsilon_{J_0}^{J^1})b=h_1\;\bullet$$

THEOREM 3.3. Assume $H\in Ob\,\aleph$. Then there exists an isomorphism

$$v:H\to Hal(relH).$$

is commutative for the corresponding homomorphisms v and v_1 .

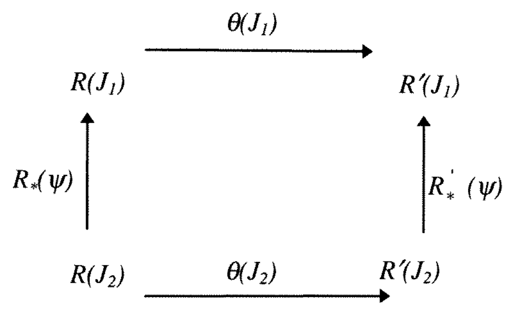
PROOF. Assume $h\in H$ and let $h\in H(J)$ for the arbitrary $J\in\Delta(h)$. We have

$$f'v(h)=\mu_1(J)f\mu^{-1}(J)\mu(J)(h)=\mu_1(J)f(h)=v_1f(h)\bullet$$

3.2. Relations between homomorphisms.

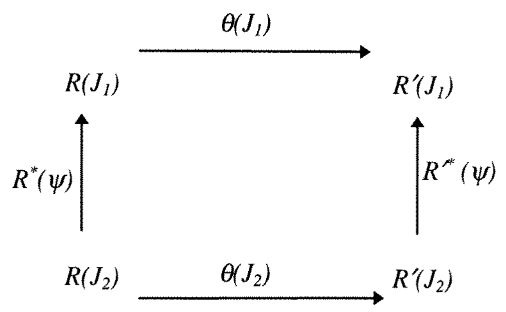
As a conclusion of the chapter we examine a question of relations between several kinds of the homomorphisms of HA-s and RA-s. Let us fix the category $\mathfrak{R}$ of RA-s over the scheme category K and the category $\aleph$ of the Halmos Algebras over the scheme $n:I\to\Gamma$. Consider $R,R'\in Ob\mathfrak{R}$. Remember that the system of the mappings $\theta = \{\theta(J)\,|\,n_J:J\to\Gamma\in ObK\}$ is a homomorphism of the RA-s, if:

1. $\theta(J):R(J)\to R'(J)$ is a homomorphism of BA-s;
2. For any morphism $\psi:J_1\to J_2$ the diagram



is commutative,

3. For any morphism $\psi:J_1\to J_2$ the diagram



is commutative.

DEFINITION 3.1 A homomorphism $\theta: R\text{->}R'$ of the $RA\text{-}s$ is called **insufficient** if conditions 1 and 2 hold and the third is true only for the injective morphisms $\psi: J_1 \text{->} J_2$.

DEFINITION 3.2. A homomorphism $\theta: R\text{->}R'$ is called **essentially insufficient** if only the first and second conditions hold.

DEFINITION 3.3. A homomorphism $\theta: R\text{->}R'$ is called **boolean** if only condition 1 takes place, while the second holds only for the injective morphisms.

Now consider $H, H' \in Ob\aleph$. Let us get, like above, definitions of several homomorpisms of H and H'.

DEFINITION 3.4. A homomorphism $\mu: H\text{->}H'$ is called **insufficient** if μ is a homomorpism of Boolean Algebras H and H', besides μ preserves existential quantifiers and elements of S_I (so, μ does not preserve an identity).

DEFINITION 3.5. A homomorphism $\mu: H\text{->}H'$ is called **essentially insufficient** if μ is a boolean homomorphism and μ preserves the elements of S_I.

DEFINITION 3.6. A homomorphism $\mu: H\text{->}H'$ is called **boolean** if μ is a homomorphism of Boolean Algebras H and H'.

COROLLARY 3.1.

I. Suppose that H_1, $H_2 \in Ob\aleph$ and let $relH_1$ and $relH_1$ be two corresponding Relational Algebras. Then:

1.To an insufficient homomorphism of $HA\text{-}s$ H_1 and H_2 there corresponds an insufficient homomorphism of $RA\text{-}s$ $relH_1$ and $relH_2$;

2.To an essentially insufficient homomorphism of $HA\text{-}s$ H_1 and H_2 there corresponds an essentially insufficient homomorphism of $RA\text{-}s$ $relH_1$ and $relH_2$;

3.To a boolean homomorphism of $HA\text{-}s$ H_1 and H_2 there corresponds a homomorphism of $RA\text{-}s$ $relH_1$ and $relH_2$;

II. Suppose that R_1, $R_2 \in Ob\Re$ and let $HalR_1$ and $HalR_1$ be the corresponding Halmos Algebras. Then:

1.To an insufficient homomorphism of $RA\text{-}s$ R_1 and R_2 there corresponds an insufficient homomorphism of $HA\text{-}s$ $HalH_1$ and $HalH_2$;

2.To an essentially insufficient homomorphism of $RA\text{-}s$ R_1 and R_2 there corresponds an essentially insufficient homomorphism of $HA\text{-}s$ $HalR_1$ and $HalR_2$;

3.To a boolean homomorphism of $RA\text{-}s$ R_1 and R_2 there corresponds a boolean homomorphism of $HA\text{-}s$ $HalR_1$ and $HalR_2$..

APPENDIX

ABSTRACT. We would like to consider two results in this Appendix. The first of them is devoted to the proof of facts, which are connected with different structures RA from the category $\mathfrak{R}$.

The second one was proved by the author together with V.Sustavova and this theorem is one of the several results from [18]. We will show here that the fifth axiom of the Relational Algebras in the sense of Beniaminov may be generalized, i.e. it may be presented like the second axiom in the definition of the Relational Algebra which was given by B.Plotkin (see [13] and chapter I). Note, however, that definitions of the RA-s which were given by E.Beniaminov and B.Plotkin, respectively, were considered over different categories-schemes (we mean the pure RA-s).

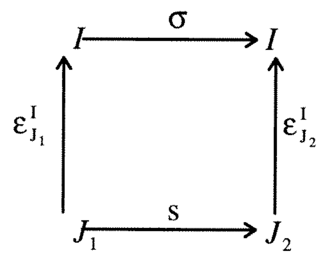
Let us make some remarks before theorem I. According to theorem 2.3 of the main text of the present work any Boolean Algebra *R(J)* of arbitrary Relational Algebra from the category $\mathfrak{R}$ may be presented as Halmos Algebra over the corresponding scheme $n_J : J \to \Gamma$. Thus, using the functor *Hal* we can again transform this Halmos Algebra *R(J)* over the scheme $n_J : J \to \Gamma$ to the Relational Algebra. In particular, using the means of the Halmos Algebra *R(J)* (which were originally created by the means of RA *R)* we can define the operations s^* and s_* for $s \in S_J$. Moreover we must get the same results.

THEOREM I. Assume that $n_{J_1} : J_1 \to \Gamma$, $n_{J_2} : J_2 \to \Gamma$ are any objects such that $J_1 \cap J_2 = \varnothing$, $J_1 = \{\alpha_1, \ldots, \alpha_2\}$, and let $s: J_1 \to J_2$ be a morphism. Then for every $a \in R(J_2)$ the expression

$$s^* a = (\varepsilon^I_{J_1})^* [(\varepsilon^I_{J_2})_* a \cap (s^{s\alpha_1}_{\alpha_1})^* 1 \cap \ldots \cap s^{s\alpha_n}_{\alpha_n})^* 1],$$

holds. Here $I = J_1 \cup J_2$, 1 is the unit of BA *R(I)*, $s^{s\alpha_i}_{\alpha_i} \in S_I, i = 1, \ldots, n$.

PROOF. Note the obvious commutativity of the following diagram



i.e. $\varepsilon^I_{J_2} \cdot s\alpha = \sigma \cdot \varepsilon^I_{J_1}\alpha, \quad \alpha \in J_1$, where $\sigma\alpha = s\alpha$, $\alpha \in J_1$ and if $\sigma\alpha = \alpha$, $\alpha \in I \setminus J$, this implies

$$s^* (\varepsilon^I_{J_2})^* b = (\varepsilon^I_{J_1})^* \sigma^* b, \quad b \in R(I).$$

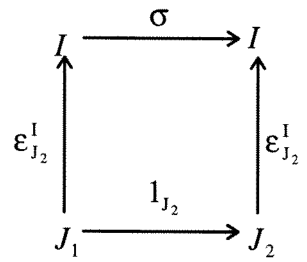
So for any $a \in R(J_2)$ we have

$$s^* a = s^* (\varepsilon^I_{J_2})^* (\varepsilon^I_{J_2})_* a = (\varepsilon^I_{J_1})^* \sigma^* (\varepsilon^I_{J_2})_* a .$$

Let us denote $(\varepsilon^I_{J_2})_* a = b$ and $(s^{s\alpha_1}_{\alpha_1})^* 1 \cap \ldots \cap s^{s\alpha_n}_{\alpha_n})^* 1 = D$, where 1 is the unit of BA *R(I)*.

Note that $$\sigma^* b = \sigma_* (\varepsilon^I_{J_2})_* a = (\sigma \varepsilon^I_{J_2})_* a = (\varepsilon^I_{J_2})_* a = b,$$

because of the commutativity of the simple diagram



i.e. $\sigma\cdot\varepsilon^{I}_{J_2}\alpha=\varepsilon^{I}_{J_2}\cdot 1_{J_2}\alpha,\alpha\in J_2$. Now using the axioms of the identity and of RA we get

$$\sigma_*(b\cap D)=b \text{ and } \sigma^*\sigma_*(b\cap D)\subseteq b\cap D\Rightarrow\sigma^*b\subseteq b\cap D.$$

Thus for $a\in R(J_1)$

$$s^*a=(\varepsilon^{I}_{J_1})^*\sigma^*(\varepsilon^{I}_{J_2})_*a\subseteq(\varepsilon^{I}_{J_1})^*[(\varepsilon^{I}_{J_2})_*a\cap(s^{s\alpha_1}_{\alpha_1})^*1\cap\ldots\cap(s^{s\alpha_n}_{\alpha_n})^*1].$$

Let us prove the inverse inclusion. It is clear that $\sigma^*b\subseteq\sigma^*b\cap D$. On the other hand we have

$$\sigma^*b=\sigma^*(b\cap\sigma_*D)\subseteq\sigma^*(\sigma_*\sigma^*b\cap\sigma_*D)=\sigma^*\sigma_*(\sigma^*b\cap D)\subseteq\sigma_*b\cap D,$$

and this implies $\sigma^*b=\sigma^*b\cap D.$ Then

$$\sigma^*b=\sigma^*b\cap D=(s^{s\alpha_1}_{\alpha_1})_*\sigma^*b\cap D=(s^{s\alpha_2}_{\alpha_2})_*(s^{s\alpha_1}_{\alpha_1})_*\sigma^*b\cap D=(s^{s\alpha_2}_{\alpha_2}\cdot s^{s\alpha_1}_{\alpha_1})_*\sigma^*b\cap$$

$$\cap D=\ldots=(s^{s\alpha_n}_{\alpha_n}\ldots s^{s\alpha_2}_{\alpha_2}\cdot s^{s\alpha_1}_{\alpha_1})_*\sigma^*b\cap D=\sigma_*\sigma^*b\cap D\supseteq b\cap D,$$

i.e. $\sigma^*b\supseteq b\cap D$. Here we used the following obvious property of the equality

$$c\cap(s^{s\alpha_i}_{\alpha_i})^*1=(s^{s\alpha_i}_{\alpha_i})_*c\cap(s^{s\alpha_i}_{\alpha_i})^*1,$$

where $c\in R(I),s^{s\alpha_1}_{\alpha_1}\in S_I,i=1,\ldots,n.$

So we get

$$s^*a=(\varepsilon^{I}_{J_1})^*\sigma^*(\varepsilon^{I}_{J_2})_*a\supseteq(\varepsilon^{I}_{J_1})^*[(\varepsilon^{I}_{J_2})_*a\cap(s^{s\alpha_1}_{\alpha_1})^*1\cap\ldots\cap(s^{s\alpha_n}_{\alpha_n})^*1]$$

and finally

$$s^*a=(\varepsilon^{I}_{J_1})^*[(\varepsilon^{I}_{J_2})_*a\cap(s^{s\alpha_1}_{\alpha_1})^*1\cap\ldots\cap(s^{s\alpha_n}_{\alpha_n})^*1]$$

as desired •

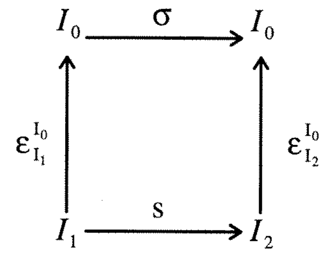
THEOREM II. Assume that $s_1\colon J_1\to J_3$, $s_2\colon J_2\to J_4$ are the morphisms such that $J_1\cap J_2=\varnothing$, $J_3\cap J_4=\varnothing$. Let us denote $I_1=J_1\cup J_2$, $I_2=J_3\cup J_4$ and consider the morphism $s\colon I_1\to I_2$, where $s\alpha=s_1\alpha$ for $\alpha\in J_1$ and $s\alpha=s_2\alpha$ for $\alpha\in J_2$. Then for every $a\in R(J_3)$, $a\in R(J_4)$ the following equalities hold:

$$s^*(a\times b)=s_1a\times s_2b,$$
$$s^*[(\varepsilon^{I_2}_{J_3})_*a\cap(\varepsilon^{I_2}_{J_4})_*b]=(\varepsilon^{I_1}_{J_1})_*s_1{}^*a\cap(\varepsilon^{I_1}_{J_2})_*s_2{}^*b.$$

PROOF. CASE I. Assume that $J_1\cap J_2\cap J_3\cap J_4=\varnothing$. Let $J_1=\{\alpha_1,\ldots,\alpha_n\}$, $J_2=\{\beta_1,\ldots,\beta_k\}$ and denote $I_0=I_1\cup I_2$. Then

$$s^*[(\varepsilon^{I_2}_{J_3})_*a\cap(\varepsilon^{I_2}_{J_4})_*b]=s^*(\varepsilon^{I_0}_{I_2})^*(\varepsilon^{I_0}_{I_2})_*[(\varepsilon^{I_2}_{J_3})_*a\cap(\varepsilon^{I_2}_{J_4})_*b].$$

Let us consider the commutative diagram



where $\sigma\alpha = s\alpha$ for $\alpha \in I_1$ and $\sigma\alpha = \alpha$ $\alpha \in I_1 \setminus I_1$. It's commutativity gives

$$s^*(\varepsilon_{I_2}^{I_0})^* c = (\varepsilon_{I_2}^{I_0})^* \sigma^* c, \quad c \in R(J_0).$$

Then

$$s^*(\varepsilon_{I_2}^{I_0})^*(\varepsilon_{I_2}^{I_0})_*[(\varepsilon_{J_3}^{I_2})_* a \cap (\varepsilon_{J_4}^{I_2})_* b] = (\varepsilon_{I_1}^{I_0})^* \sigma^*[(\varepsilon_{I_2}^{I_0})_*(\varepsilon_{J_3}^{I_2})_* a \cap$$
$$\cap (\varepsilon_{I_2}^{I_0})_*(\varepsilon_{J_4}^{I_2})_* b] = (\varepsilon_{I_1}^{I_0})^* \sigma^*[(\varepsilon_{J_3}^{I_0})_* a \cap (\varepsilon_{J_4}^{I_0})_* b] =$$
$$= (\varepsilon_{I_1}^{I_0})^*[(\varepsilon_{J_3}^{I_0})_* a \cap (\varepsilon_{J_4}^{I_0})_* b \cap (s_{\alpha_1}^{s_1\alpha_1})^* 1_{R(I_0)} \cap$$
$$\cap \ldots \cap (s_{\alpha_n}^{s_1\alpha_n})^* 1_{R(I_0)} \cap (s_{\beta_1}^{s_2\beta_1})^* 1_{R(I_0)} \cap \ldots \cap (s_{\beta_k}^{s_2\beta_k})^* 1_{R(I_0)}].$$

Here we used the result of theorem 1 of the Appendix. Denote

$$D(J_1) = (s_{\alpha_1}^{s_1\alpha_1})^* 1_{R(I_0)} \cap \ldots \cap (s_{\alpha_n}^{s_1\alpha_n})^* 1_{R(I_0)},$$
$$D(J_2) = (s_{\beta_1}^{s_2\beta_1})^* 1_{R(I_0)} \cap \ldots \cap (s_{\beta_k}^{s_2\beta_k})^* 1_{R(I_0)},$$
$$D = D(J_1) \cap D(J_2).$$

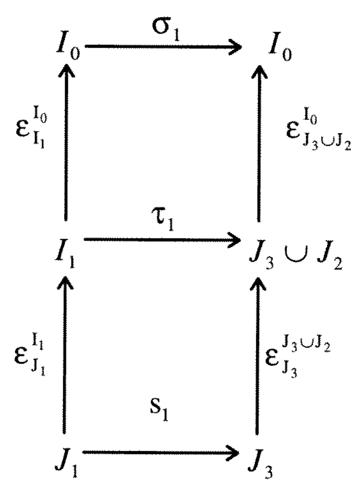
Using theorem 1 once more we obtain

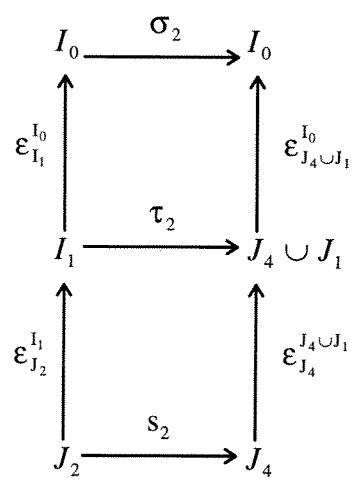
$$(\varepsilon_{I_1}^{I_0})^*[(\varepsilon_{J_3}^{I_0})_* a \cap (\varepsilon_{I_4}^{I_0})_* b \cap D] = (\varepsilon_{I_1}^{I_0})^*[(\varepsilon_{J_3}^{I_0})_* a \cap D(J_1) \cap$$
$$\cap (\varepsilon_{I_4}^{I_0})_* b \cap D(J_2)] = (\varepsilon_{I_1}^{I_0})^*[\sigma_1^*(\varepsilon_{J_3}^{I_0})_* a \cap \sigma_2^*(\varepsilon_{I_4}^{I_0})_* b]$$

where $\sigma_1, \sigma_2 \in S_{I_0}$ and $\sigma_1\alpha = s_1\alpha$ if $\alpha \in J_1$ and $\sigma_1\alpha = \alpha$ if $\alpha \in I_0 \setminus J_1$; $\sigma_2\alpha = s_2\alpha$ if $\alpha \in J_2$ and $\sigma_2\alpha = \alpha$ if $\alpha \in I_0 \setminus J_2$. Then we get the following expressions:

$$(\varepsilon_{J_3}^{I_0})\alpha = (\varepsilon_{J_3 \cup J_2}^{I_0} \cdot \varepsilon_{J_3}^{J_3 \cup J_2})\alpha, \alpha \in J_3,$$
$$(\varepsilon_{J_4}^{I_0})\beta = (\varepsilon_{J_4 \cup J_4}^{I_0} \cdot \varepsilon_{J_4}^{J_4 \cup J_1})\beta, \beta \in J_4.$$

Consider two diagrams





where $\tau_1\alpha = s_1\alpha$ if $\alpha \in J_1$ and $\tau_1\alpha = \alpha$ if $\alpha \in J_2$; $\tau_2\alpha = s_2\alpha$ if $\alpha \in J_2$ and $\alpha \in J_2$ $\tau_2\alpha = \alpha$ if $\alpha \in J_1$. It is easy to see that both these diagrams are commutative. Using this we get the commutativity of further diagrams

$$\begin{array}{ccc} R(I_0) & \xleftarrow{\sigma_1^*} & R(I_0) \\ \uparrow (\varepsilon_{I_1}^{I_0})_* & & \uparrow (\varepsilon_{J_3\cup J_2}^{I_0})_* \\ R(I_1) & \xleftarrow{\tau_1^*} & R(J_3 \cup J_2) \\ \uparrow (\varepsilon_{J_1}^{I_1})_* & & \uparrow (\varepsilon_{J_3}^{J_3\cup J_2})_* \\ R(J_1) & \xleftarrow{s_1^*} & R(J_3) \end{array} \qquad \begin{array}{ccc} R(I_0) & \xleftarrow{\sigma_2^*} & R(I_0) \\ \uparrow (\varepsilon_{I_1}^{I_0})_* & & \uparrow (\varepsilon_{J_4\cup J_1}^{I_0})_* \\ R(I_1) & \xleftarrow{\tau_2^*} & R(J_4 \cup J_1) \\ \uparrow (\varepsilon_{J_2}^{I_1})_* & & \uparrow (\varepsilon_{J_4}^{J_4\cup J_1})_* \\ R(J_2) & \xleftarrow{s_2^*} & R(J_4) \end{array}$$

Commutativity of these diagrams follows from lemma 1.3. So we get

$$\sigma_1^*(\varepsilon_{J_3}^{I_0})_* a = \sigma_1^*(\varepsilon_{J_3\cup J_2}^{I_0} \cdot \varepsilon_{J_3}^{J_3\cup J_2})_* a = \sigma_1^*(\varepsilon_{J_3\cup J_2}^{I_0})_*(\varepsilon_{J_3}^{J_3\cup J_2})_* a =$$
$$= (\varepsilon_{I_1}^{I_0})_* \tau^*{}_1(\varepsilon_{J_3}^{J_3\cup J_2})_* a = (\varepsilon_{I_1}^{I_0})_*(\varepsilon_{J_1}^{I_1})_* s_1^* a, \quad a \in R(J_3);$$
$$\sigma_2^*(\varepsilon_{J_4}^{I_0})_* b = \sigma_2^*(\varepsilon_{J_4\cup J_1}^{I_0} \cdot \varepsilon_{J_4}^{J_4\cup J_1})_* b = \sigma_2^*(\varepsilon_{J_4\cup J_1}^{I_0})_*(\varepsilon_{J_4}^{J_4\cup J_1})_* b =$$
$$= (\varepsilon_{I_1}^{I_0})_* \tau_2^*(\varepsilon_{J_4}^{J_4\cup J_1})_* b = (\varepsilon_{I_1}^{I_0})_*(\varepsilon_{J_2}^{I_1})_* s_2^* b, \quad b \in R(J_4).$$

Finally we have

$$s^*(a \times b) = s^*[(\varepsilon_{J_3}^{I_2})_* a \cap (\varepsilon_{J_4}^{I_2})_* b] = (\varepsilon_{I_1}^{I_0})^*[\sigma_1^*(\varepsilon_{J_3}^{I_0})_* a \cap \sigma_2^*(\varepsilon_{I_4}^{I_0})_* b] =$$
$$= (\varepsilon_{I_1}^{I_0})^*(\varepsilon_{I_1}^{I_0})_*[(\varepsilon_{J_1}^{I_1})_* \cdot s_1^* a \cap (\varepsilon_{J_2}^{I_1})_* \cdot s_2^* b] = (\varepsilon_{J_1}^{I_1})_* s_1^* a \cap (\varepsilon_{J_2}^{I_1})_* s_2^* b =$$
$$= s_1^* a \times s_2^* b.$$

Thus the proof of case I is completed.

CASE II. $J_1 \cap J_2 = \varnothing$, $J_3 \cap J_4 = \varnothing$. Take into consideration two objects $n_{J_1'}: J_1' \to \Gamma$, $n_{J_2'}: J_2' \to \Gamma$ such that $J_1' \cap J_2' = \varnothing$, besides both J_1' and J_2' are mutually non-intersected with the sets J_1, J_2, J_3 and J_4 and there exist two bijections $s_1': J_1 \to J_1'$, $s_2': J_2 \to J_2'$.

So we have two commutative diagrams

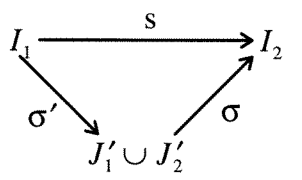
$$\begin{array}{ccc} J_1 & \xrightarrow{s_1} & J_3 \\ & {\scriptstyle s_1'}\searrow \quad \nearrow{\scriptstyle s_1''} & \\ & J_1' & \end{array} \qquad \begin{array}{ccc} J_2 & \xrightarrow{s_1} & J_4 \\ & {\scriptstyle s_2'}\searrow \quad \nearrow{\scriptstyle s_2''} & \\ & J_2' & \end{array}$$

where s_1'' and s_2'' are the morphisms which are constructed in a natural way such that $s_1\alpha = s_1''s_1'\alpha, \alpha \in J_1$ and $s_2\alpha = s_2''s_2'\alpha, \alpha \in J_2$.. Now let us construct two morphisms $\sigma': I_1 \to J_1' \cup J_2'$ and $\sigma: J_1' \cup J_2' \to I_2$ (remember that $I_1 = J_1 \cup J_2$, $I_2 = J_3 \cup J_4$) by the following rules :

$a. \sigma'\alpha = s_1'\alpha$ if $\alpha \in J_1$, $\sigma'\alpha = s_2'\alpha$ if $\alpha \in J_2$;

b. $\sigma\alpha = s_1''\alpha$ if $\alpha \in J_1'$, $\sigma\alpha = s_2''\alpha$ if $\alpha \in J_2''$.

Thus we have one more commutative diagram



i.e. $s\alpha = \sigma\sigma'\alpha$, $\alpha \in I_1$. And now using the previously examined case I of the theorem, we have for every $h \in R(J_1')$, $g \in R(J_2')$, $a \in R(J_3)$, $b \in R(J_4)$

$$\sigma'^*(h \times g) = s_1'^* h \times s_2'^* g,\ \sigma^*(a \times b) = s_1''^* a \times s_2''^* b.$$

Denote $h = s_1''^* a$ and $g = s_2''^* b$. Then

$$s_1'^* h \times s_2'^* g = s_1'^* s_1''^* a \times s_2'^* s_2''^* b = (s_1'' s_1')^* a \times (s_2'' s_2')^* b = s_1' a \times s_2' b.$$

On the other hand

$$s_1'^* h \times s_2'^* g = \sigma'^*(h \times g) = \sigma'^*(s_1''^* a \times s_2''^* b) = \sigma'^*(\sigma^*(a \times b)) = (\sigma\sigma')^*(a \times b) =$$
$$= s^*(a \times b) \bullet$$

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